Bounds on the Hausdorff Measure of Level-N Sierpinski Gaskets

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Abstract

Although a favorite of fractal geometers, the Hausdorff measure of many classical fractals is often difficult to calculate or even bound. In this work we review some important definitions and results from fractal geometry and define the fractal known as the level-N Sierpinski gasket. By generalizing a previous technique used for the classical Sierpinski gasket, the main result of this work obtains an upper bounds for the Hausdorff measure of the level-N Sierpinski gasket.

1 Introduction

Fractal sets often show the same pattern at multiple scales. Many prototypical examples of fractal sets can be generated by what are known as iterated function systems—a finite collection of contraction maps on the same space. The classical Cantor set and Sierpinski gasket are examples of fractals that arise from iterated function systems.

Fractals have many fascinating properties; among them are their dimension and measure. While fractal dimensions have been carefully explored, much is not known about the measure of fractal sets; see [Fal14], [Hut81]. A common approach to measuring the size of a fractal set is the use of Hausdorff measure, a generalization of the Lebesgue measure. The Hausdorff measure approximates the size of a set with sets of sufficiently small diameters. It also includes a parameter $s \geq 0$ which grants flexibility in how a set grows as it is scaled. For sets like lines, planes, or cubes the parameter s is an integer and reflects the topological dimension of the set being measured. For the fractal sets we consider in this work the parameter s will give a non-integer dimension.

We now state some preliminary definitions before formally defining the Hausdorff measure and Hausdorff dimension of a set in $\mathbb{R}^n$.

Definition 1.1. Let $F \subseteq \mathbb{R}^n$. A collection $\{U_i\}_{i\in I}$ of subsets of $\mathbb{R}^n$ is a **cover** of F if

$$F \subseteq \bigcup_{i\in I} U_i.$$

The **diameter** of a set F is defined by

$$\operatorname{diam}(F) = \sup\{|x-y| : x, y \in F\}.$$

Definition 1.2. Let $F \subseteq \mathbb{R}^n$, $s \geq 0$, and $\delta > 0$. Define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s : F \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } \operatorname{diam}(U_i) < \delta\right\}. \quad (1)$$

As δ decreases, the infimum is taken over a reduced class of permissible covers of F so $\mathcal{H}^s_\delta(F)$ increases. The limit

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$$

is called the **s-dimensional Hausdorff measure of F**. The **Hausdorff dimension** of F is defined to be

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}.$$

We will expand upon the definition of Hausdorff dimension. If $r > s$ we have for a δ cover $\{U_j\}_{j\in J}$ of F that

$$\sum_{j\in J} \operatorname{diam}(U_j)^r = \sum_{j\in J} \operatorname{diam}(U_j)^{r-s} \operatorname{diam}(U_j)^s \leq \delta^{r-s} \sum_{j\in J} \operatorname{diam}(U_j)^s$$

and hence $\mathcal{H}^r_\delta(F) \leq \delta^{r-s}\mathcal{H}^s_\delta(F)$.

Letting $\delta \to 0$ we see that if for some s, $\mathcal{H}^s(F) < \infty$ then for all $r > s$, $\mathcal{H}^r(F) = 0$. This means that for most values of s the Hausdorff measure of a fixed set $F \subseteq \mathbb{R}^n$ is either 0 or infinity. The value of s at which the Hausdorff measure of a set switches from being infinite to being 0, gives us a notion of dimension that is better suited for studying fractal sets than the typical topological dimension; see Figure 1.

For even the classical fractal sets, the calculation of Hausdorff measure can be quite involved. The original intent of this work was to investigate and improve upon the calculation of the Hausdorff measure of the Sierpinski gasket (Figure 2). In the process, we investigated a more general class of fractals and we present the results from that work here.

We now briefly summarize the progress on bounding the Hausdorff measure of the Sierpinski gasket, $\mathcal{S}$. The reader should note that for $\mathcal{S}$, the Hausdorff dimension is $\dim_H(\mathcal{S}) = \log_2(3)$.

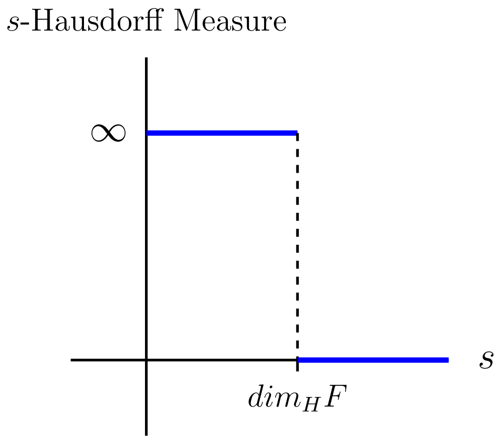

Figure 1: Graph of $\mathcal{H}^s(F)$ against s for a fixed set $F \subseteq \mathbb{R}^n$.

Upper bound on $\mathcal{H}^s(\mathcal{S})$:

- (1987) $\mathcal{H}^s(\mathcal{S}) \leq 0.9508$, [Mar87]
- (1997) $\mathcal{H}^s(\mathcal{S}) \leq 0.9105$, [Zho97a]
- (1997) $\mathcal{H}^s(\mathcal{S}) \leq 0.8900$, [Zho97b]
- (1999) $\mathcal{H}^s(\mathcal{S}) \leq 0.8180$, [WW99] (In Chinese)
- (2000) $\mathcal{H}^s(\mathcal{S}) \leq 0.8308$, [ZF00]

Lower bound of $\mathcal{H}^s(\mathcal{S})$:

- (2002) $\mathcal{H}^s(\mathcal{S}) \geq 0.5000$, [JZZ02]
- (2004) $\mathcal{H}^s(\mathcal{S}) \geq 0.5631$, [HW04]
- (2006) $\mathcal{H}^s(\mathcal{S}) \geq 0.6704$, [JZZ06]
- (2009) $\mathcal{H}^s(\mathcal{S}) \geq 0.7700$, [Mór09]

Our work is based on the results of Zhou in 1997. We note that these are not the best known methods for calculating an upper bound for the Hausdorff measure of the Sierpinski gasket; however, Zhou's method is readily available for extension to a larger family of level-N Sierpinski gaskets. The main result of our work is the following theorem found in Section 3.

Theorem. For the class of level-N Sierpinski gaskets, Zhou's method may be extended to show that

$$\mathcal{H}^s(\mathcal{S}^N) \leq \min_{m\in\mathbb{N}} \left(1 + \frac{3}{\left(\frac{N(1+N)}{2}\right)^m - 5}\right)\left(1 - \frac{1}{N^m - 1}\right)^s.$$

We now give a brief summary of the sections that follow:

- Section 2 introduces a number of important definitions and helpful results regarding Hausdorff measure.
- Section 3 introduces a fractal known as the level-N Sierpinski gasket. We then present the main result of this paper—an extension of the method in [Zho97b] which yields an upper bound on the Hausdorff measure of the level-N Sierpinski gasket for any $N \geq 2$.
- Section 4 summarises our results and makes some closing remarks.

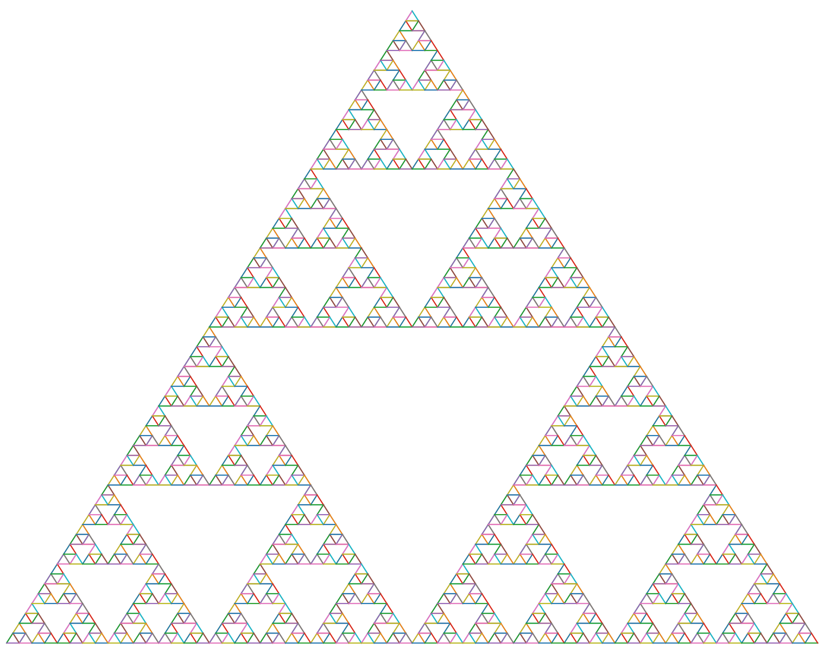

Figure 2: Sierpinski gasket

2 Key Definitions

One property shared by many fractals is that of self-similarity. One may observe that fractals such as the Sierpinski gasket are made up of pieces that are geometrically similar to the entire set; see Figure 2. Iterated function systems can be used to construct many interesting fractal sets with rich structure and a simple method of calculating the Hausdorff dimension. We now state a number of definitions which will be essential to the remaining sections.

Definition 2.1. A **contraction** is a mapping $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$|S(x) - S(y)| \leq r|x - y|$$

with $0 < r < 1$ for all $x, y \in \mathbb{R}^n$. The number r is the **ratio** of the contraction S. If equality holds above, then we say S is a **similarity** which transforms subsets of $\mathbb{R}^n$ into geometrically similar sets. A finite family of contractions $\{S_1, S_2, \ldots, S_m\}$, where $S_i : \mathbb{R}^n \to \mathbb{R}^n$, is called an **iterated function system** or IFS.

The following theorem tells us how to associate a set to an IFS. The proof of the theorem is an application of the Banach fixed point theorem; a complete proof can be found in [Fal14, pp. 135–136].

Theorem 2.1. Let $\{S_i\}_{i=1}^m$ be a collection of contractions on $D \subset \mathbb{R}^n$. Then there exists a unique non-empty compact set $F \subset \mathbb{R}^n$ such that

$$F = \bigcup_{i=1}^{m} S_i(F).$$

This set F is called the **attractor** or **invariant set** of the IFS $\{S_i\}_{i=1}^m$.

The attractor of an IFS made up of a collection of similarities is called a *self-similar set*. With the addition of a separation property, we may calculate the Hausdorff dimension of a self-similar set using a theorem found in [Fal14, p. 140]. This separation condition is known as the *open set condition* and requires the existence of a non-empty bounded open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V$$

where this union is disjoint.

Theorem 2.2. Let S_i be similarities on $\mathbb{R}^n$ with ratios r_i $(1 \leq i \leq m)$ and which satisfy the open set condition. If F is the set satisfying

$$F = \bigcup_{i=1}^{m} S_i(F)$$

then $\dim_H(F) = s$ where s is the real number satisfying

$$\sum_{i=1}^{m} r_i^s = 1.$$

The Cantor set for example may be defined as the attractor of the following IFS,

$$\begin{cases} S_1(x) = \frac{1}{3}x \\ S_2(x) = \frac{1}{3}x + \frac{2}{3}. \end{cases}$$

Thus, it is easy to calculate that the Hausdorff dimension of the Cantor set is $\log_3(2)$. As another example, the Sierpinski gasket is the attractor for the IFS

$$F_i(x) = \frac{1}{2}(x - p_i) + p_i$$

where $p_1 = (0,0)$, $p_2 = (1,0)$ and $p_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Since each of these 3 similarities has ratio $\frac{1}{2}$, we see that the Hausdorff dimension of the Sierpinski gasket is $\dim_H(\mathcal{S}) = \log_2(3)$.

3 The Level-N Sierpinski Gasket

3.1 Definition of $\mathcal{S}^N$

We next introduce a class of fractals which generalizes the construction of the Sierpinski gasket. For an integer $N \geq 2$, we informally define the level-N Sierpinski Gasket, $\mathcal{S}^N$, by the following process; see Figure 3 and Figure 4 for example.

1. Begin with a unit triangle with vertices (0,0),(1,0), $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and denote it S_0^N
2. In the next iteration, S_1^N is formed by dividing S_0^N into N^2 equal triangles that are each a scaled copy of S_0^N by a factor of $\frac{1}{N}$.
3. To reach S_{k+1}^N, repeat the division described in step 2 on each upright triangle of S_k^N
4. $\mathcal{S}^N$ is obtained by taking $k \to \infty$ above.

Formally, the level-N Sierpinski gasket is defined as the attractor of the IFS given by the similarities produced by the following algorithm:

```
for i from 0 to N − 1
    for j from 0 to N − i
        f_{i,j}(x) = 1/N [x + i(1/2, √3/2) + j(1,0)]
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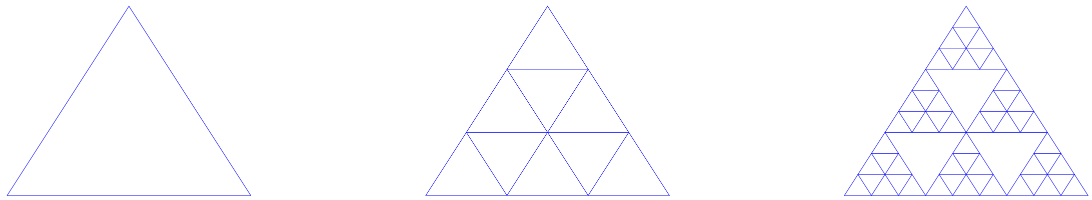
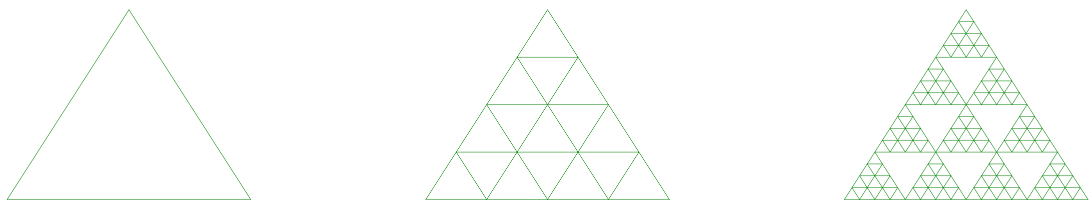


(a) S_0^3 (b) S_1^3 (c) S_2^3

Figure 3: Construction of $\mathcal{S}^3$



(a) S_0^4 (b) S_1^4 (c) S_2^4

Figure 4: Construction of $\mathcal{S}^4$

For example, when $N = 3$ we have the following IFS:

$$\begin{cases} f_{0,0}(x) = \frac{1}{3}x \\ f_{0,1}(x) = \frac{1}{3}\left(x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ f_{0,2}(x) = \frac{1}{3}\left(x + \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) \end{cases} \qquad \begin{cases} f_{1,0}(x) = \frac{1}{3}\left(x + \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}\right) \\ f_{1,1}(x) = \frac{1}{3}\left(x + \begin{bmatrix} 3/2 \\ \sqrt{3}/2 \end{bmatrix}\right) \\ f_{2,0}(x) = \frac{1}{3}\left(x + \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}\right) \end{cases}$$

which yields the level-3 Sierpinski gasket. Note that the classic Sierpinski gasket, $\mathcal{S}$, is given by the case $N = 2$. Also, it is easy to see that there are $\frac{1}{2}N(N+1)$ similarities for the N case. By Theorem 2.2, a calculation will yield that the level-N Sierpinski gasket has Hausdorff dimension

$$\dim_H(\mathcal{S}^N) = \frac{\log\left(\frac{N(N+1)}{2}\right)}{\log(N)}. \tag{2}$$

3.2 Extension of Zhou's Method

Here we present the main results of this paper. In [Zho97b], Zhou introduces a method which establishes upper bounds for the Hausdorff measure of the classical Sierpinski gasket. We take Zhou's construction and extend it to sets $\mathcal{S}^N$.

Let $f_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity transformation of ratio $\alpha > 0$. If $F \subset \mathbb{R}^2$, then by [Fal14, p. 46], we have the scaling property

$$\mathcal{H}^s(f_\alpha(F)) = \alpha^s \mathcal{H}^s(F). \tag{3}$$

For convenience, we'll define $f_\alpha^n = f_\alpha \circ f_\alpha^{n-1}$ for $n = 2, 3, 4, \dots$, with $f_\alpha^1 = f_\alpha$.

For the remainder of this work we will set $s = \log_N \left(\frac{N(N+1)}{2}\right)$. Let $\beta = \{U_i : i \in \mathbb{N}\}$ be a δ-cover of $\mathcal{S}^N$ and let $\epsilon_\beta \in \mathbb{R}$ be the error in using β to estimate $\mathcal{H}^s(\mathcal{S}^N)$, then

$$\mathcal{H}^s(\mathcal{S}^N) = \sum_{i=0}^{\infty} \operatorname{diam}(U_i)^s + \epsilon_\beta.$$

Let $k \in \mathbb{N}$. Compressing β via $f_{1/N}^k$ results in a δ/N^k-cover of $f_{1/N}^k(\mathcal{S}^N)$. Taking $\left(\frac{N(N+1)}{2}\right)^k$ duplicates of $f_{1/N}^k(\beta)$ reconciles a δ/N^k-cover of $\mathcal{S}^N$ by exploiting the set's symmetry. The reader should note that

$$\left(\frac{N(N+1)}{2}\right)^k = N^{sk}$$

where $s = \log_N \left(\frac{N(N+1)}{2}\right)$. With this in mind, we consider the following proposition.

Proposition 3.1. We have

$$\mathcal{H}^s(\mathcal{S}^N) = \left(\frac{N(N+1)}{2}\right)^k \sum_{i=1}^{\infty} \left(\frac{1}{N^k} \operatorname{diam}(U_i)\right)^s + \epsilon_\beta = \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s + \epsilon_\beta \tag{4}$$

and

$$\mathcal{H}^s\left(f_{1/N}^k(\mathcal{S}^N)\right) = \left(\frac{N(N+1)}{2}\right)^{-k} \mathcal{H}^s(\mathcal{S}^N) = \sum_{i=1}^{\infty} \left(\frac{\operatorname{diam}(U_i)}{N^k}\right)^s + \left(\frac{N(N+1)}{2}\right)^{-k} \epsilon_\beta. \tag{5}$$

Proof. Equation (4) is clear. Equation (5) follows from (3) and (4). ■

Proposition 3.2. Let $\delta > 0$. Then

$$\mathcal{H}^s(\mathcal{S}^N) = \mathcal{H}^s_\delta(\mathcal{S}^N). \tag{6}$$

Proof. We first note that $\mathcal{H}^s(\mathcal{S}^N) \geq \mathcal{H}^s_\delta(\mathcal{S}^N)$ is clear by definition. Let $\epsilon > 0$. Since

$$\mathcal{H}^s_\delta(\mathcal{S}^N) = \inf\left\{\sum_{i=1}^{\infty} \text{diam}(A_i)^s : \{A_i\} \text{ is a } \delta\text{-cover of } \mathcal{S}^N\right\},$$

there exists a δ-cover, $\Gamma = \{V_i : i \in \mathbb{N}\}$, such that

$$\sum_{i=1}^{\infty} \text{diam}(V_i) \leq \mathcal{H}^s_\delta(\mathcal{S}^N) + \epsilon.$$

Letting $k \geq 0$ and using N^{sk} duplicates of $f^k_{1/N}(\Gamma)$ as a $\frac{\delta}{N^k}$-cover of $\mathcal{S}^N$, we have

$$\mathcal{H}^s_{\delta/N^k}(\mathcal{S}^N) \leq N^{sk} \sum_{i=1}^{\infty} \left(\frac{1}{N^k} \text{diam}(V_i)\right)^s = \sum_{i=1}^{\infty} \text{diam}(V_i)^s \leq \mathcal{H}^s_\delta(\mathcal{S}^N) + \epsilon.$$

Taking $k \to \infty$ and noting that ϵ is free gives us that

$$\mathcal{H}^s(\mathcal{S}^N) \leq \mathcal{H}^s_\delta(\mathcal{S}^N).$$

■

Our focus is on the level-N Sierpinski gaskets, but one should note that these propositions may be extended to the class of all self-similar fractal sets. Now we move to the main result of the paper.

Theorem 3.3. For the class of level-N Sierpinski gaskets Zhou's method [Zho97b] may be extended to show that

$$\mathcal{H}^s(\mathcal{S}^N) \leq \min_{m \in \mathbb{N}} \left(1 + \frac{3}{\left(\frac{N(1+N)}{2}\right)^m - 5}\right)\left(1 - \frac{1}{N^m - 1}\right)^s. \tag{7}$$

Proof. Our goal is to build a sequence of covers of $\mathcal{S}^N$ whose diameters are decreasing. In what follows we will denote by S^N_i, the i-th step in the construction of the level-N Sierpinski gasket. Beginning from S^N_0, we label the 3 vertices $A = (0, 0)$, $B = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $C = (1, 0)$. We iterate $m \in \mathbb{N}$ times to S^N_m and label 6 points,

$$\begin{cases} a^m_1 : \left(\frac{1}{2N^m}, \frac{\sqrt{3}}{2N^m}\right) \\ a^m_2 : \left(\frac{1}{N^m}, 0\right) \\ b^m_1 : \left(\frac{N^m-1}{2N^m}, \frac{\sqrt{3}(N^m-1)}{2N^m}\right) \\ b^m_2 : \left(\frac{N^m+1}{2N^m}, \frac{\sqrt{3}(N^m-1)}{2N^m}\right) \\ c^m_1 : \left(\frac{N^m-1}{N^m}, 0\right) \\ c^m_2 : \left(\frac{2N^m-1}{2N^m}, \frac{\sqrt{3}}{2N^m}\right) \end{cases}$$

which are the pairs of vertices from stage m which are nearest to the vertices A, B, and C; see Figure 5a. We form three triangles using A, B, C and their corresponding pairs:

$$\triangle A a_1^m a_2^m, \qquad \triangle B b_1^m b_2^m, \qquad \triangle C c_1^m c_2^m.$$

For convenience, we denote the set of points contained in a single one of these triangles by T_N^m. Thus, by taking three copies of T_N^m, denoted 3-T_N^m, and positioning them correctly, we recover $\triangle A a_1^m a_2^m, \triangle B b_1^m b_2^m$ and $\triangle C c_1^m c_2^m$.

Next, we use these 6 neighbors again to form a hexagon and label the set of points contained in it H_N^m; see Figure 6a.

Note that $\operatorname{diam}(H_N^m) = 1 - \frac{1}{N^m}$. The union of H_N^m with $3(2^0)$ copies of T_N^m forms a cover of $\mathcal{S}^N$, which we label σ_1; see Figure 6a,

$$\sigma_1 = \{H_N^m,\ 3(2^0)\text{-}T_N^m\}.$$

To reach σ_2, we iterate m times again to reach S_{2m}^N. From each of a_1^m, a_2^m, b_1^m, b_2^m, c_1^m, and c_2^m, we mark the pair of nearest neighbors closest to the center of S_{2m}^N; see Figure 5b. Label the triangles formed as T_N^{2m}, of which there are $3(2^1)$, and we shrink our hexagon H_N^m slightly such that these new neighbors lie on its boundary; see Figure 6b.

We label this hexagon H_N^{2m} and note that $\operatorname{diam}(H_N^{2m}) = 1 - \left(\frac{1}{N^m} + \frac{1}{N^{2m}}\right)$. Now we define

$$\sigma_2 = \{H_N^{2m},\ 3(2^0)\text{-}T_N^m,\ 3(2^1)\text{-}T_N^{2m}\}.$$

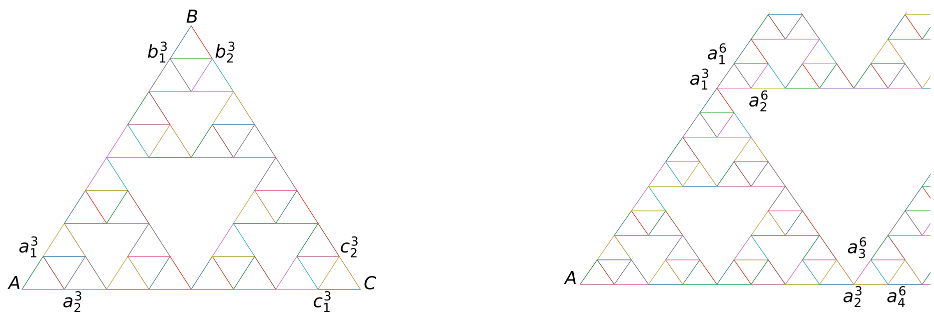
Continuing in this process, we have, at the n^{th}-step,

$$\sigma_n = \{H_N^{nm},\ 3(2^0)\text{-}T_N^m,\ 3(2^1)\text{-}T_N^{2m},\ \ldots,\ 3(2^{n-1})\text{-}T_N^{nm}\}$$

where

$$\operatorname{diam}(H_N^{nm}) = 1 - \sum_{i=1}^{n} \frac{1}{N^{mi}}.$$

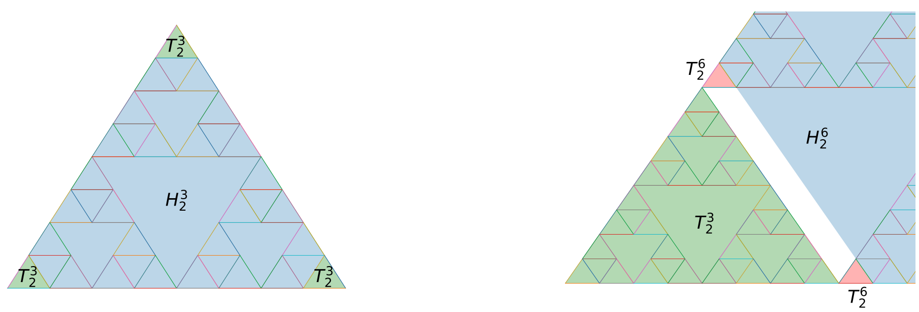
Thus, we have a sequence of coverings of $\mathcal{S}^N$, $\{\sigma_n\}_{n=1}^{\infty}$. We will further modify this cover to obtain the result we want.



(a) Points forming σ_1.

(b) Points used in the left corner formation of σ_2

Figure 5: Zhou's Method on $\mathcal{S}$ with $m = 3$ (Points).



(a) Visualization of σ_1.

(b) Visualization of the left corner of σ_2.

Figure 6: Zhou's Method on $\mathcal{S}$ with $m = 3$ (Sets).

Let $n \geq 0$ and let $\beta = \{V_i : i \in \mathbb{N}\}$ be a cover of $\mathcal{S}^N$. For convenience, let us use $f_{1/N}^{nm}(\beta)$ to denote the collection of sets $f_{1/N}^{nm}(V_i)$ where $i \in \mathbb{N}$. In other words,

$$f_{1/N}^{nm}(\beta) = \{f_{1/N}^{nm}(V_i) : i \in \mathbb{N}\}$$

By definition, $f_{1/N}^{nm}(\beta)$ covers $f_{1/N}^{nm}(\mathcal{S}^N)$; after perhaps a translation of the sets, we may assume that $f_{1/N}^{nm}(\beta)$ covers $f_{1/N}^{nm}(\mathcal{S}^N) \cap T_N^{nm}$. Thus, we may replace T_N^{nm} with $f_{1/N}^{nm}(\beta)$ in σ_n for $n = 1, 2, 3, \ldots$, and obtain a new sequence of covers, $\{\upsilon_n\}_{n=1}^{\infty}$, where

$$\upsilon_n = \{H_N^{nm},\ 3(2^0)\text{-}f_{1/N}^{m}(\beta),\ 3(2^1)\text{-}f_{1/N}^{2m}(\beta),\ \ldots,\ 3(2^{n-1})\text{-}f_{1/N}^{nm}(\beta)\}.$$

By Propositions 3.1-3.2 we have

$$\mathcal{H}^s(\mathcal{S}^N) = \mathcal{H}^s_1(\mathcal{S}^N) \tag{8}$$

$$\leq \left(1 - \sum_{k=1}^{n} \frac{1}{N^{mk}}\right)^s + 3\left(\sum_{j=1}^{n} \frac{2^{j-1}}{\left(\frac{N(N+1)}{2}\right)^{mj}}\right) \sum_{i=1}^{\infty} \mathrm{diam}(V_i)^s \tag{9}$$

where in particular we have used the fact that

$$\mathrm{diam}(f_{1/N}^{nm}(V_i))^s = N^{-nms}\,\mathrm{diam}(V_i)^s = \left(\frac{N(N+1)}{2}\right)^{-nm} \mathrm{diam}(V_i)^s.$$

Using the error ϵ_β and Proposition 3.1,

$$\mathcal{H}^s(\mathcal{S}^N) \leq \left(1 - \sum_{k=1}^{n} \frac{1}{N^{mk}}\right)^s + 3\left(\sum_{j=1}^{n} \frac{2^{j-1}}{\left(\frac{N(N+1)}{2}\right)^{mj}}\right) \left(\mathcal{H}^s(\mathcal{S}^N) - \epsilon_\beta\right). \tag{10}$$

Noting that we may choose β so that $|\epsilon_\beta|$ is sufficiently small, we have that

$$\mathcal{H}^s(\mathcal{S}^N) \leq \left(1 - \sum_{k=1}^{n} \frac{1}{N^{mk}}\right)^s + 3\left(\sum_{j=1}^{n} \frac{2^{j-1}}{\left(\frac{N(N+1)}{2}\right)^{mj}}\right) \mathcal{H}^s(\mathcal{S}^N). \tag{11}$$

Evaluating terms, we have

$$\mathcal{H}^s(\mathcal{S}^N) \leq \left(\frac{N^m - 2 + N^{-nm}}{N^m - 1}\right)^s + \frac{3}{\left(\frac{N(N+1)}{2}\right)^m - 2}\left[1 - \left(\frac{2^{m+1}}{N^m(N+1)^m}\right)^n\right] \mathcal{H}^s(\mathcal{S}^N).$$

Thus,

$$\mathcal{H}^s(\mathcal{S}^N) \leq \frac{\left(\frac{N^m-2}{N^m-1} + \frac{N^{-nm}}{N^m-1}\right)^s}{1 - \frac{3}{\left(\frac{N(N+1)}{2}\right)^m - 2}\left[1 - \left(\frac{2^{m+1}}{N^m(N+1)^m}\right)^n\right]} \tag{12}$$

The term on the right of (12) is decreasing as $n \to \infty$. One can justify this by noting that the numerator will decrease as n gets larger while the denominator tends to $1 - 3\left(\frac{1}{\left(\frac{N(N+1)}{2}\right)^m - 2}\right)$ as n gets larger. We take the limit $n \to \infty$ and obtain

$$\mathcal{H}^s(\mathcal{S}^N) \leq \frac{\left(\frac{N^m-2}{N^m-1}\right)^s}{1 - 3\left(\frac{1}{\left(\frac{N(1+N)}{2}\right)^m - 2}\right)}$$

$$= \left(1 + \frac{3}{\left(\frac{N(1+N)}{2}\right)^m - 5}\right)\left(1 - \frac{1}{N^m - 1}\right)^s.$$

Finally, we simply need to minimize over $m \in \mathbb{N}$, which gives us the desired result in (7). ■

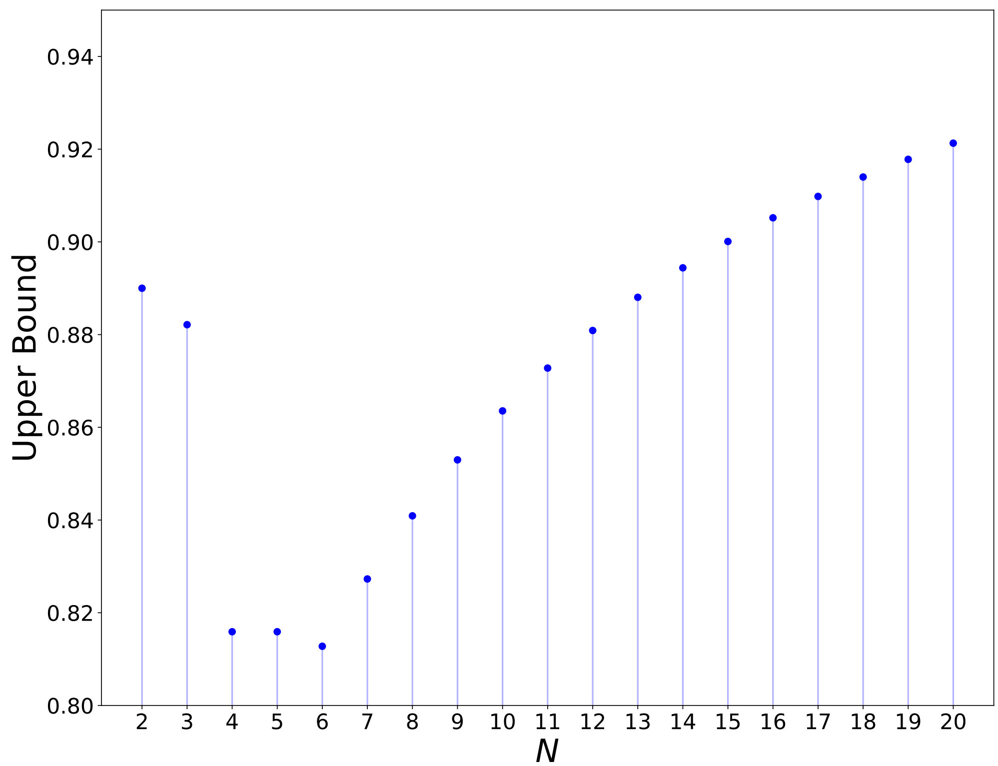


Figure 7: Upper bounds of $\mathcal{H}^s(\mathcal{S}^N)$ vs. N

4 Closing Remark

Below we show a summarizing table of the correspondence between N and m for $N = 1, \ldots, 6$. See also Figure 7 for a graph which plots the upper bound for $\mathcal{H}^s(\mathcal{S}^N)$ for levels 2 through 20.

N	2	3	4	5	6
$m_{\min}$	3	2	1	1	1
$\mathcal{H}^s(\mathcal{S}^N) \leq$	0.890039	0.882138	0.815903	0.801163	0.812767

In [ZF00], Zhou and Feng extend the method of approximating the Hausdorff measure of $\mathcal{S}$ by using hexagons to a method which instead uses dodecagons. In the future, we would like to investigate how these improved methods can yield better approximations for the Hausdorff measure of the level-N Sierpinski gaskets as well as other fractal sets.

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