UCR Differential Equations

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Assistance from:

Contents

1 Important Theory:

Gronwall Inequality (Brauer Thm. 1.4)

- **Theorem:** Let K be a nonnegative constant and let $f, g : [\alpha, \beta] \to \mathbb{R}$ be continuous nonnegative functions satisfying

$$f(t) \leq K + \int_{\alpha}^{t} f(s)g(s)ds$$

for $\alpha \leq t \leq \beta$. Then

$$f(t) \leq K \exp\left\{\int_{\alpha}^{t} g(s)ds\right\}$$

for all $t \in [\alpha, \beta]$.

Proof Outline.

1. Set $u(t) := K + \int_{\alpha}^{t} f(s)g(s)ds$
2. Take $u'(t)$ and use the fact that $f(t) \leq u(t)$
3. Force the product rule by multiplying an integrating factor.
4. Integrate from α to t.
5. Move things around and note that $f(t) \leq u(t)$.

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First Existence and Uniqueness (Brauer Thm. 1.1)

- **Theorem:** Let F be a vector function (with n components) defined in a region D of $\mathbb{R}^{n+1}$. Let the vectors F and $\partial F/\partial y_k$ be continuous in D for all $k = 1, \ldots, n$. Then given a point $(t_0, \eta) \in D$, there exists a unique continuous solution ϕ of the system

$$y' = f(t, y) \qquad y(t_0) = \eta$$

The solution ϕ exists on an interval I containing t_0 for which the points $(t, \phi(t)) \in D$ when $t \in I$.

Linear System Existence and Uniqueness (Brauer Thm. 2.1)

- **Theorem:** If $A(t), g(t)$ are continuous on some interval $a \leq t \leq b$, if $a \leq t_0 \leq b$, and if $|\eta| < \infty$, then the system $y' = A(t)y + g(t)$ has a unique solution $\phi(t)$ satisfying $\phi(t_0) = \eta$ and ϕ exists on $a \leq t \leq b$.

- Note that the interval for which the solution ultimately exists on depends on domain in which $F(t, y) = A(t)y + g(t)$ is continuous. If $D = \text{dom}(F)$ is given by

$$D = [a, b] \times (-\infty, \infty)$$

then the existence interval, which proliferates from t_0 continues so long as $|\phi(t)| < \infty$, i.e. for $t \in [a, b]$, the point $(t, \phi(t))$ remains in D.

Abel's Formula (Brauer Thm. 2.3)

- **Theorem:** If Φ is a solution matrix of

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$$

on I and if $t_0 \in I$, then

$$\det \Phi(t) = \det \Phi(t_0) \exp\left\{\int_{t_0}^{t} \sum_{j=1}^{n} a_{jj}(s) ds\right\} \qquad t \in I$$

Fundamental Matrix Criteria (Brauer Thm. 2.4)

- **Definition:** A solution matrix on I for $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ whose columns are linearly independent on I is called a *fundamental matrix*.

- **Theorem:** A solution matrix Φ of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ on an interval I is a fundamental matrix on I iff $\det \Phi(t) \neq 0$ for all $t \in I$.

Variation of Constants Formula (Brauer Thm. 2.6)

- **Theorem:** If Φ is a fundamental matrix of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ on an interval I, then

$$\Psi(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)g(s)ds$$

is the unique solution of

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t)$$

satisfying $\Phi(t_0) = \eta$.

- Using this, we have that any solution to $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t)$ can be written as

$$\mathbf{y}(t) = \Phi_h(t) + \Psi(t)$$

where Φ is as stated above and Φ_h is the solution to the homogeneous equation such that the initial conditions agree.

Fundamental Matrix for Constant Coefficient Linear System (Brauer Thm. 2.7)

- **Theorem:** The matrix
$$\Phi(t) = e^{At}$$
is the fundamental matrix of $y' = Ay$ with $\Phi(0) = I_n$ on $-\infty < t < \infty$.

- If A is a constant coefficient matrix, then the solution to the system
$$\begin{cases} \mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t) \\ \mathbf{y}(0) = \eta \end{cases}$$
is given by
$$\mathbf{y}(t) = e^{\mathbf{A}t}\eta + \int_0^t e^{\mathbf{A}(t-s)} g(s) ds$$

Eigenvalue bound on Fundamental Matrix (Brauer Thm. 2.10)

- **Theorem:** If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A, where λ_j has multiplicty n_j and $n_1 + \cdots + n_k = n$ and if p is any number larger than the real part of $\lambda_1, \ldots, \lambda_k$, i.e.
$$p > \max_{j=1,\ldots,k} \Re(\lambda_j)$$
then there exists a constant $K > 0$ such that
$$|\exp\{t\mathbf{A}\}| \leq K \exp\{pt\} \qquad t \in [0, \infty)$$

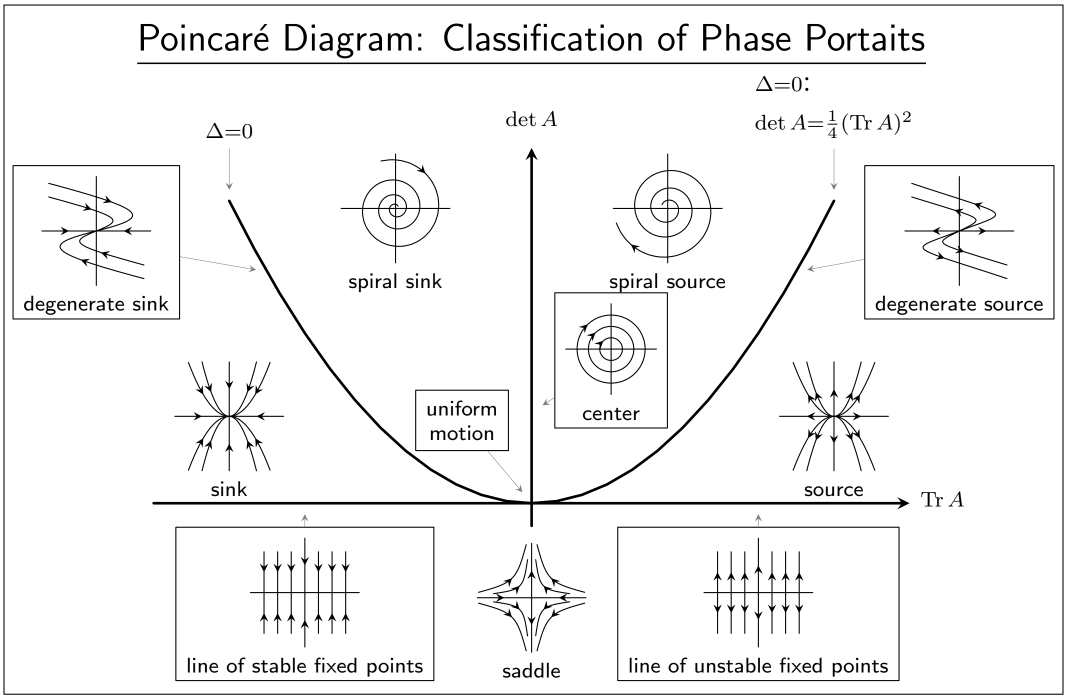
Existence Theorem (Brauer Thm. 3.1)

- The system we will situate ourselves in is
$$y' = f(t, y) \qquad y(t_0) = y_0$$
with $f, \partial f/\partial y$ continuous on the rectangle R given by
$$R = \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\}$$

- **Lemma:** Define α to be the smaller of the positive numbers $a, b/\|f\|_\infty$. Then the successive approximations ϕ_n given by
$$\begin{cases} \phi_0(t) = y_0 \\ \phi_{n+1}(t) = y_0 + \int_{t_0}^t f(s, \phi_n(s)) ds \quad n = 1, 2, \ldots \end{cases}$$
is well defined on the interval $I = \{t : |t - t_0| \leq \alpha\}$ and on this interval
$$|\phi_n(t) - y_0| \leq \|f\|_\infty |t - t_0| \leq b \qquad n = 1, 2, \ldots$$

- **Theorem:** Suppose $f, \partial f/\partial y$ are continuous on the closed rectangle R. Then the successive approximations ϕ_n, converge uniformly on the interval I to a solution ϕ of the above system.

Poincare Diagram: Phase Portrait Classification



Bifurcation Normal Forms (Strogatz Ch. 3)

Each type of bifurcation has a prototypical *normal form*.

1. (Saddle-node)
$$x' = r + x^2$$
2. (Transcritical)
$$x' = rx - x^2$$
3. (Supercritical pitchfork)
$$x' = rx - x^3$$
4. (Subcritical pitchfork)
$$x' = rx + x^3$$

Fundamental Solution of Laplace's Equation (Evans Sec. 2.2.1)

- **Definition:** The function

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & (n \geq 3) \end{cases}$$

  defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the fundamental solution of Laplace's equation, $\Delta u = 0$.

- We also have the following estimates on the gradient and Hessian of Φ,

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \qquad |\Delta^2\Phi(x)| \leq \frac{C}{|x|^n} \qquad (x \neq 0)$$

  for some $C > 0$.

Mean Value Formula for Harmonic Functions (Evans. Thm 2.2.2)

- **Theorem:** If $u \in C^2(\Omega)$ is harmonic, then

$$u(x) = \fint_{\partial B_r(x)} u(y)dS(y) = \fint_{B_r(x)} u(y)dS(y)$$

  for each ball $B_r(x) \subset \Omega$.

  Proof Outline.

  1. Define a function $\phi(r) = \fint_{\partial B_r(x)} u(y)dS(y)$.
  2. Use a change of coordinates so that we're integrating over $\partial\Omega$. This is $y \mapsto x + rz$ $(dS(z))$ and a factor of r^{n-1} appears as well so that we preserve the average.
  3. Take $\phi'(r)$ so that a z pops out and convert back to y so that the z becomes $\frac{y-x}{r}$ which is exactly the unit normal vector.
  4. Use Green's theorem so convert the integral to a useful formula, $\phi'(r) = \frac{r}{n}\fint_{B_r(x)} \Delta u(y)dy$ and use harmonicity.
  5. Thus, ϕ is constant so we can take $r \to 0$ to get $u(x)$.
  6. For $\fint_{B_r(x)}$, use polar coordinates to pull out $\fint_{\partial B_r(x)}$ and use the mean value formula over the surface.

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- **Theorem:** If $u \in C^2(\Omega)$ satisfies

$$u(x) = \fint_{\partial B_r(x)} u(y)dS(y)$$

  for each ball $B_r(x) \subset \Omega$, then u is harmonic.

Proof Outline.

1. Suppose $\Delta u(x_0) > 0$.
2. Define $\phi(r) = \fint_{\partial B_r(x_0)} u(y)dS(y)$, then we still get $\phi'(r) = \frac{r}{n} \fint_{B_r(x_0)} \Delta u(y)dy$.
3. The hypothesis gives us that $\phi(r) = u(x_0)$ for every r, so ϕ is constant which leads to the contradiction.

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Strong Maximum Principle for Laplace's Equation (Evans Thm. 2.2.4)

- **Theorem:** Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic within Ω. Then,

1. $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.
2. If Ω is connected and there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u,$$

then u is constant in Ω.

Proof Outline.

1. Proving (2) first, if $x_0 \in \Omega$ is maximal, then draw the ball $B_{\text{dist}(x_0,\partial\Omega)}(x_0)$ and use the mean value formula.
2. Thus, $B_{\text{dist}(x_0,\partial\Omega)}(x_0) \subset u^{-1}(\{u(x_0)\})$ which shows openness of $u^{-1}(\{u(x_0)\})$. Closedness of $u^{-1}(\{u(x_0)\})$ follows from $\{u(x_0)\}$ being a singleton, hence closed (preimage of closed is closed). Thus, it must be the entire set Ω.
3. Then use connectedness and that u is continuous to $\partial\Omega$.
4. To show (1), just use the same assumption and we'll get u constant on an open component of Ω. Then take u continuous to $\partial\Omega$ for the contradiction.

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Uniqueness of Solution to Poisson's Equation (Evans Thm. 2.2.5)

- **Theorem:** Let $g \in C(\partial\Omega)$, $f \in C(\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Smoothness of Harmonic Functions (Evans Thm. 2.2.6)

- **Theorem:** If $u \in C(\Omega)$ satisfies the mean value property for each ball $B_r(x) \subseteq \Omega$, then
$$u \in C^\infty(\Omega)$$

Proof Outline.

1. Let η be the standard mollifier which we note is radial and define $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$ which has $\operatorname{supp}(\eta_\epsilon) \subset B_\epsilon(0)$.
2. Set $u^\epsilon = \eta_\epsilon * u$ in $\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$ and we know u^ϵ is smooth.
3. Calculate using the definition of η_ϵ, polar coordinates, and the mean value property to get that $u^\epsilon(x) = u(x)$ in Ω_ϵ for all ϵ.
4. Conclude that $u \in C^\infty(\Omega)$.

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Harnack's Inequality for Harmonic Functions (Evans Thm. 2.2.11)

- **Theorem:** For each connected open set V with $V \subset\subset \Omega$, there exists a positive constant C, depending only on V, such that
$$\sup_V u \leq C \inf_V u$$
for all nonnegative harmonic functions u in Ω.

Proof Outline.

1. Let $r := \frac{1}{4}\operatorname{dist}(V, \partial\Omega)$ and choose $x, y \in V$ with $|x - y| < r$
2. Use mean value formula over $B_{2r}(x)$, u nonnegative, and $B_r(y) \subset B_{2r}(x)$ to calculate $u(x) \geq \frac{1}{2^n}u(y)$.
3. Use V connected, $\overline{V}$ compact to cover $\overline{V}$ be a finite chain of overlapping balls of radius $r/2$.
4. Induct over the number of balls and repeat (2) to get $u(x) \geq \frac{1}{2^{n(N+1)}u(y)}$ for any $x, y \in V$.

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Poisson's Formula for the Ball (Evans Thm. 2.2.15)

- **Theorem:** If $u \in C^2(\overline{\Omega})$ solves Poisson's equation,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

for $f \in C(\Omega)$, $g \in C(\partial\Omega)$, then

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu}(x,y) dS(y) + \int_{\Omega} f(y) G(x,y) dy \qquad (x \in \Omega)$$

- **Definition:** Green's function for the unit ball is

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y - \tilde{x})) \qquad (x, y \in B_1(0), x \neq y)$$

where $\tilde{x} = \frac{x}{|x|^2}$.

- **Theorem:** Assume $g \in C(\partial B_r(0))$ and define u by

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} dS(y) + \underbrace{\int_{B_r(0)} f(y) G(x,y) dy}_{\text{Inhomogeneous term}}$$

then

  - $u \in C^\infty(B_r(0))$.
  - $\Delta u = 0$ in $B_r(0)$
  - $\lim_{\substack{x \to x_0 \\ x \in B_r(0)}} u(x) = g(x_0)$ for each point $x_0 \in \partial B_r(0)$.

Energy Method for Uniqueness of Poisson's (Evans Thm. 2.2.16)

- **Theorem:** There exists at most one solution $u \in C^2(\overline{\Omega})$ of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof Outline.

1. Consider two solutions u_1, u_2 satisfying the above equation and take their difference $w = u_1 - u_2$.
2. We then see $\Delta w = 0$ and $w = 0$ on $\partial\Omega$, so integrate $w\Delta w$ by parts to find $|Dw| = 0$

3. Hence $w = 0$ in Ω.

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Dirichlet's Principle (Evans Thm. 2.2.17)

- **Theorem:** Assume $u \in C^2(\overline{\Omega})$ solves

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Then,

$$I[u] = \min_{w \in \mathcal{A}} I[w] \qquad \text{where} \qquad \begin{cases} I[w] := \int_\Omega \frac{1}{2}|Dw|^2 - wf dy \\ \mathcal{A} := \{w \in C^2(\overline{\Omega}) : w = g \text{ on } \partial\Omega\} \end{cases}$$

Conversely, if $u \in A$, satisfies the above minimization problem, then u solves the Poisson equation above.

Proof Outline.

1. (Forward direction) First notice that $0 = \int_\Omega (-\Delta u - f)(u - w)dy$ since $-\Delta u - f = 0$.
2. Distribute and integrate $-\Delta u(u-w)$ by parts. Moving things around gives $\int_\Omega |Du|^2 - fudy = \int_\Omega Du \cdot Dw - fw$.
3. Using the Cauchy Schwarz and Cauchy's inequality, we know $|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2$
4. Use (2) on $\int_\Omega Du \cdot Dw - fw$ to find $I[w]$ and move things around to get $I[u] \leq I[w]$
5. (Backward direction) Consider a small perturbation $i(\epsilon) := I[u + \epsilon v]$ where $\epsilon \in \mathbb{R}$ and $v \in C_c^\infty(\Omega)$.
6. Note that $i'(0) = 0$ since $\epsilon = 0$ is minimal
7. Expand and distribute $i(\epsilon)$, take $\frac{d}{d\epsilon}$ of $i(\epsilon)$ and set $\epsilon = 0$.
8. Integrate by parts to find $0 = \int_\Omega (-\Delta u - f)vdy$
9. Since this holds for every $v \in C_c^\infty(\Omega)$, then $-\Delta - f = 0$.

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Fundamental Solution of the Heat Equation (Evans Sec. 2.3.1)

- **Definition:** The function

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases}$$

  is called the fundamental solution of the heat equation, $u_t - \Delta u = 0$.

- **Lemma:** (Integral of fundamental solution). For each time $t > 0$,

$$\int_{\mathbb{R}^n} \Phi(x,t)dx = 1.$$

  Note the choice of normalizing constant makes this possible.

Inhomogeneous Initial Value Heat Equation (Evans Thm. 2.3.2)

- **Theorem:** Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and define u by

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)dyds \\ &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy + \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)dyds \end{aligned}$$

  for $x \in \mathbb{R}^n$, $t > 0$, then

  1. $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$.
  2. $u_t(x,t) - \Delta u(x,t) = f(x,t)$ for $x \in \mathbb{R}^n, t > 0$.
  3. $\lim_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}^n, t>0}} u(x,t) = g(x_0)$ for each $x_0 \in \mathbb{R}^n$.

Mean Value Formula for the Heat Equation (Evans Thm. 2.3.3)

- **Definition:** We define the parabolic cylinder

$$\Omega_T := \Omega \times (0, T]$$

and the parabolic boundary of Ω_T is

$$\Gamma_T := \overline{\Omega}_T - \Omega_T$$

Be careful to note that Ω_T contains the interior and the top face while Γ_T comprises the bottom face and the vertical sides.

- **Definition:** For fixed $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $r > 0$, we define

$$E(x, t; r) := \left\{ (y, s) \in \mathbb{R}^{n+1} : s \leq t,\ \ \Phi(x - y, t - s) \geq \frac{1}{r^n} \right\}$$

Note that the "center" (x, t) is located at the center of the top of the heat ball.

- **Theorem:** Let $u \in C_1^2(\Omega_T)$ solve the heat equation. then

$$u(x, t) = \frac{1}{4r^n} \iint_{E(x,t;r)} u(y, s) \frac{|x - y|^2}{(t - s)^2} dyds$$

for each $E(x, t; r) \subset \Omega_T$.

Strong Maximum Principle for Heat Equation (Evans Thm. 2.3.4)

- **Theorem:** Assume $U \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ solves the heat equation in Ω_T. Then

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u$$

Furthermore, if Ω is connected and there exists a point $(x_0, t_0) \in \Omega_T$ such that

$$u(x_0, t_0) = \max_{\overline{\Omega}_T} u$$

then u is constant in $\overline{\Omega}_{t_0}$.

Uniqueness of Solution to Heat Equation (Evans Thm. 2.3.5)

- **Theorem:** Let $g \in C(\Gamma_T)$, $f \in C(\Omega_T)$. Then there exists at most one solution

$u \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ of the initial/boundary-value problem

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Smoothness of Solution to the Heat Equation (Evans Thm. 2.3.8)

- **Theorem:** Suppose $u \in C_1^2(\Omega_T)$ solves the heat equation in Ω_T. Then $u \in C^\infty(\Omega_T)$.

Energy Method for Uniqueness of Heat Equation (Evans Thm. 2.3.10)

- **Theorem:** (Forward uniqueness) There exists only one solution $u \in C_1^2(\overline{\Omega}_T)$ of the initial/boundary-value problem.

$$\begin{cases} u_t - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Proof. Let u_1, u_2 be solutions to the heat equation and define $w := u_1 - u_2$ so that w solves

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \Omega_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

Set

$$E(t) = \int_{\Omega_T} \frac{1}{2} w^2(x,t)dx \qquad 0 \le t \le T$$

Taking ∂_t, we have

$$\begin{aligned} E'(t) &= \int_{\Omega_T} w(x,t)w_t(x,t)dx \\ &= \int_{\Omega_T} w(x,t)\Delta w(x,t)dx && \text{(by the PDE)} \\ &= -\int_{\Omega_T} |Dw|^2 dx && \text{(int. by parts)} \\ &\le 0 \end{aligned}$$

Therefore, $E(t) \le E(0) = 0$ since $w = 0$ on Γ_T. Thus, $u_1 - u_2 = w = 0$ in Ω_T. □

- **Theorem:** (Backwards uniqueness) Suppose $u_1, u_2 \in C^2(\overline{\Omega}_T)$ solve

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = g & \text{on } \partial\Omega \times [0,T] \end{cases}$$

If $u_1(x,T) = u_2(x,T)$ for $x \in \Omega$, then $u_1 = u_2$ in Ω_T.

Proof. Let u_1, u_2 be solutions to the heat equation and define $w := u_1 - u_2$ so that w solves the homogeneous heat equation with zero boundary condition on Γ_T. Set

$$E(t) = \int_{\Omega_T} \frac{1}{2} w^2(x,t)dx \qquad 0 \leq t \leq T$$

and take ∂_t as well as ∂_t^2.

$$\begin{aligned}
E'(t) &= -\int_{\Omega_T} |Dw|^2 dx \\
E''(t) &= -2\int_{\Omega_T} Dw \cdot (Dw)_t dx \\
&= 2\int_{\Omega_T} \Delta w w_t dx && \text{(int. by parts)} \\
&= 2\int_{\Omega_T} (\Delta w)^2 dx && \text{(By the PDE)}
\end{aligned}$$

Now observe that

$$\begin{aligned}
E'(t) &= -\int_{\Omega_T} |Dw|^2 dx \\
&= -\int_{\Omega_T} w \Delta w dx && \text{(int. by parts)} \\
&\leq \|w\|_{L^2(\Omega_T)} \|\Delta w\|_{L^2(\Omega_T)}
\end{aligned}$$

Thus,

$$[E'(t)]^2 \leq \frac{1}{2} 2 \int_{\Omega_T} w^2 dx \int_{\Omega_T} (\Delta w)^2 dx = E(t)E''(t)$$

Now if $E \equiv 0$ for all $t \in [0,T]$, then we are done, so assume otherwise so that there exists an interval $[t_1, t_2] \subset [0,T]$ where $E(t) > 0$ for $t \in [t_1, t_2)$ and $E(t_2) = 0$. Such a t_2 exists since we can push t_2 to T and we know that $w(x,T) = 0$ by hypothesis. Now define

$$f(t) := \log(E(t)) \qquad t \in [t_1, t_2)$$

and we see that

$$\begin{aligned}
f'(t) &= \frac{E'(t)}{E(t)} \\
f''(t) &= \frac{E(t)E''(t) - [E'(t)]^2}{[E(t)]^2} \\
&= \frac{E''(t)}{E(t)} - \frac{[E'(t)]^2}{[E(t)]^2} \\
&\geq 0 && \text{(since } [E']^2 \leq EE''\text{)}
\end{aligned}$$

Thus, f is convex, so for $\lambda \in (0,1)$ and $t \in (t_1, t_2)$

$$f(\lambda t_1 + (1-\lambda)t) \leq \lambda f(t_1) + (1-\lambda) f(t)$$

and exponentiating gives

$$0 \leq E(\lambda t_1 + (1-\lambda)t) \leq E^{\lambda}(t_1) E^{1-\lambda}(t)$$

so letting $t \to t_2$, we have that

$$0 \leq E(\lambda t_1 + (1-\lambda)t_2) \leq E^{\lambda}(t_1) E^{1-\lambda}(t_2) = 0 \qquad \text{for all } \lambda \in (0,1)$$

Thus, $E \equiv 0$ on $[t_1, t_2]$, a contradiction. Hence $E \equiv 0$ for $t \in [0, T]$, so $w = 0$ in Ω_T. □

d'Alembert's Formula (Evans Thm. 2.4.1)

- **Theorem:** (Solution of wave equation, $n = 1$) Assume $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and define u by d'Alembert's formula,

$$u(x,t) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \qquad x \in \mathbb{R}, t \geq 0$$

then

1. $u \in C^2(\mathbb{R} \times [0, \infty)$
2. $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times [0, \infty)$.
3. $\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u(x,t) = g(x^0)$ and $\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u_t(x,t) = h(x^0)$ for each point $x^0 \in \mathbb{R}$.

Uniqueness for Wave Equation (Evans Thm. 2.4.5)

- **Theorem:** Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with a smooth boundary $\partial\Omega$, and as usual, set $\Omega_T = \Omega \times (0, T]$, $\Gamma_T = \overline{\Omega}_T - \Omega_T$, where $T > 0$. Then there exists at most one solution $u \in C^2(\overline{\Omega}_T)$, solving

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega_T \\ u = g & \text{on } \Gamma_T \\ u_t = h & \text{on } \Omega \times \{t = 0\} \end{cases}$$

Proof Outline. 1. Let $w = u_1 - u_2$ where u_1, u_2 are solutions

2. Define $E(t) := \frac{1}{2}\int_{\Omega} w_t^2(x,t) + |Dw(x,t)|^2 dx$ for $0 \leq t \leq T$.
3. Take $E'(t)$ and use the PDE to get $E'(t) = 0$ for all $0 \leq t \leq T$.

■

Wave Equation Finite Propagation Speed (Evans Thm. 2.4.6)

- **Theorem:** If $u \equiv u_t \equiv 0$ on $B_{t_0}(x_0) \times \{t = 0\}$, then $u \equiv 0$ within the cone $K(x_0, t_0)$, where

$$K(x_0, t_0) := \{(x, t) := 0 \le t \le t_0, \ |x - x_0| \le t_0 - t\}$$

Proof. Define the energy function,

$$E(t) := \frac{1}{2} \int_{B_{t_0 - t}(x_0)} u_t^2(x, t) + |Du|^2(x, t) dx$$

Then,

$$\begin{aligned}
E'(t) &= \int_{B_{t_0-t}(x_0)} u_t u_{tt} + Du \cdot Du_t dx - \frac{1}{2} \int_{\partial B_{t_0-t}(x_0)} u_t^2 + |Du|^2 dS(x) \\
&\qquad \text{(polar coordinates (derivative))} \\
&= \int_{B_{t_0-t}(x_0)} u_t u_{tt} - u_t \Delta u dx \\
&\qquad + \int_{\partial B_{t_0-t}(x_0)} Du \cdot \eta u_t dS(x) - \frac{1}{2} \int_{\partial B_{t_0-t}(x_0)} u_t^2 + |Du|^2 dS(x) \\
&= 0 + \int_{\partial B_{t_0-t}(x_0)} Du \cdot \eta u_t dS(x) - \frac{1}{2} \int_{\partial B_{t_0-t}(x_0)} u_t^2 + |Du|^2 dS(x) \\
&\qquad \text{(by the PDE)} \\
&\le \frac{1}{2} \int_{\partial B_{t_0-t}(x_0)} |Du|^2 + u_t^2 dS(x) - \frac{1}{2} \int_{\partial B_{t_0-t}(x_0)} u_t^2 + |Du|^2 dS(x) \\
&\qquad \text{(Young's ineq.)} \\
&= 0
\end{aligned}$$

Thus, $E'(t) \le 0$. Since $u \equiv 0$ on $B_{t_0}(x_0) \times \{t = 0\}$ then $Du = 0$ on $B_{t_0}(x_0)$, so we must have that $E(t) \le E(0) = 0$ for $0 \le t \le t_0$. Thus, $u(x, t) = u(x_0, t_0) = 0$ for all $(x, t) \in K(x_0, t_0)$. □

Holder Space (Evans Thm. 5.2.1)

- If $u : \Omega \to \mathbb{R}$. Then we say u is Holder continuous with exponent γ if

$$|u(x) - u(y)| \le C|x - y|^\gamma \qquad (x, y \in \Omega), \gamma \in (0, 1], C \ge 0$$

Note if $\gamma > 1$, then u will be constant.

- **Definition:** If $u : \Omega \to \mathbb{R}$, $u \in C_b(\Omega)$, we write

$$\|u\|_{C(\overline{\Omega})} := \sup_{x \in \Omega} |u(x)|$$

- **Definition:** The γ^{th}-Holder seminorm of $u : \Omega \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

- **Definition:** So the γ^{th}-Holder norm of $u : \Omega \to \mathbb{R}$ is

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}$$

- **Definition:** The Holder space $C^{k,\gamma}(\overline{\Omega})$ consists of all functions $u \in C^k(\overline{\Omega})$ for which

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{\Omega})} < \infty \qquad (\alpha \text{ multiindex})$$

i.e. the space of functions that are up to k-times continuously differentiable and whose k^{th} derivatives are bounded and Holder continuous with exponent γ

- **Theorem:** Holder space, $C^{k,\gamma}(\overline{\Omega})$ is a Banach space.

Weak Derivative (Evans Sec. 5.2.1)

- **Definition:** Suppose $u, v \in L^1_{\text{loc}}(\Omega)$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, denoted

$$D^\alpha u = v$$

provided

$$\int_\Omega u D^\alpha \phi dy = (-1)^{|\alpha|} \int_\Omega v \phi dy \qquad \text{for all test functions } \phi \in C_c^\infty(\Omega)$$

- **Lemma:** If it exists, then the α^{th}-weak derivative of u is uniquely defined up to a set of measure zero.

Sobolev Space (Evans Sec. 5.2.2)

- **Definition:** The Sobolev space, denoted $W^{k,p}(\Omega)$, consists of all locally $L^1(\Omega)$ functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

- If $p = 2$, we usually write

$$H^k(\Omega) = W^{k,2}(\Omega) \qquad k = 0, 1, 2, \ldots$$

and the letter H is used since $H^k(\Omega)$ is a Hilbert space. Also, note that $H^0(\Omega) = L^2(\Omega)$.

- **Definition:** If $u \in W^{k,p}(\Omega)$, we define the Sobolev norm by

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx\right)^{1/p} & 1 \le p < \infty \\ \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = \infty \end{cases}$$

- **Definition:** We denote by $W_0^{k,p}(\Omega)$, the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$. (i.e. the limit points of $C_c^\infty(\Omega)$ using the Sobolev metric.)

- **Theorem:** For each $k =\in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

Elementary Properties of Weak Derivatives (Evans Thm. 5.2.1)

- **Theorem:** Assume $u, v \in W^{k,p}(\Omega)$, $|\alpha| \le k$. Then,

  (i) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all α, β with $|\alpha| + |\beta| \le k$.

  (ii) For each $\lambda \in \mathbb{R}$, $\lambda u + v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + v) = \lambda D^\alpha u + D^\alpha v$. i.e. weak derivatives are linear.

  (iii) If V is an open subset of Ω, then $u \in W^{k,p}(V)$.

  (iv) If $\zeta \in C_c^\infty(\Omega)$, then $\zeta u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\zeta u) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \qquad \text{(Leibniz formula)}$$

  where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$ where $\alpha! = \prod_{i=1}^{|\alpha|} \alpha_i!$

Approximations of Sobolev functions (Evans Sec. 5.3)

- **Theorem:** (Local Approximation) Assume $u \in W^{k,p}(\Omega)$ for some $1 \le p < \infty$, and set

$$u^\epsilon = \eta_\epsilon * u \qquad \text{in } \Omega_\epsilon$$

  Then,

  - $u^\epsilon \in C^\infty(\Omega_\epsilon)$ for each $\epsilon > 0$
  - $u^\epsilon \to u$ a.e. in Ω_ϵ.
  - $u^\epsilon \to u$ in $W^{k,p}_{\text{loc}}(\Omega)$ as $\epsilon \to 0$.

- **Theorem:** (Global Approximation) Assume Ω is bounded, and suppose that $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists functions $u_m \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that
$$u_m \to u \qquad \text{in } W^{k,p}(\Omega)$$
If we further have that $\partial\Omega$ is C^1, then we may take $u_m \in C^\infty(\overline{\Omega})$.

Extensions (Evans Sec. 5.4)

- **Theorem:** (Extension theorem) Assume Ω is bounded and $\partial\Omega$ is C^1. Select a bounded open set V such that $\Omega \subset\subset V$. Then there exists a bounded linear operator
$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$
such that for each $u \in W^{1,p}(\Omega)$.
  - $Eu = u$ a.e. in Ω
  - Eu has support (i.e. is nonzero) within V
  - $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$ where C depends only on p, Ω, V

Traces (Evans Sec. 5.5)

- **Theorem:** Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator
$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$
such that
  - $Tu = u\big|_{\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.
  - $\|Tu\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$.
- **Theorem:** Assume Ω is bounded and $\partial\Omega$ is C^1. Suppose further that $u \in W^{1,p}(\Omega)$. Then,
$$u \in W_0^{1,p}(\Omega) \qquad \text{iff} \qquad Tu = 0 \text{ on } \partial\Omega$$

Sobolev Inequalities (Evans Sec. 5.6)

- **Definition:** If $1 \leq p < n$ (n is our ambient dimension), the Sobolev conjugate of p is
$$p^* := \frac{np}{n-p}$$
Note that
$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \qquad p^* > p$$
- **Theorem:** (Gagliardo-Nirenberg-Sobolev inequality) Assume $1 \leq p < n$.

There exists a constant C, depending only on n and p, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } u \in C_c^1(\mathbb{R}^n)$$

- **Theorem:** (Estimates for $W^{1,p}(\Omega)$, $1 \leq p < n$) Let Ω be an open, bounded subset of $\mathbb{R}^n$ with $\partial\Omega$ C^1. Assume $1 \leq p < n$ and $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$ with
$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$
where C is a constant only depending on n, p, Ω.

- **Theorem:** (Estimates for $W_0^{1,p}(\Omega)$, $1 \leq p < n$) Assume Ω is a bounded open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < n$. Then, we have the estimate
$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$
for each $q \in [1, p^*]$, the constant C depending only on p, q, n, Ω.

- **Theorem:** (Morrey's inequality) Assume $n < p \leq \infty$. Then there exists a constant C, depending only on p and n, such that
$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$
for all $u \in C^1(\mathbb{R}^n)$, where $\gamma := 1 - n/p$.

- **Theorem:** (Estimates for $W^{1,p}$, $n < p \leq \infty$) Let Ω be a bounded, open, subset of $\mathbb{R}^n$, and suppose $\partial\Omega$ is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(\Omega)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{\Omega})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate
$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}$$
The constant C depends only on p, n, Ω.

  This theorem essentially allows us to replace a Sobolev function, $u \in W^{1,p}$ with $p > n$ with its Holder-continuous counterpart.

Sobolev Embeddings (Compactness) (Evans Sec. 5.7)

- **Definition:** Let X, Y be Banach spaces, $X \subset Y$. We say that X is compactly embedded in Y, denoted
$$X \subset\subset Y$$
provided
  - $\|u\|_Y \leq C\|u\|_X (u \in X)$ for some constant C.
  - Each bounded sequence $(u_k)_{k=1}^{\infty}$ in X is precompact in Y, i.e. boundedness in X implies a convergent subsequence to a limit in Y.

- **Theorem:** (Rellich-Kondrachov compactness theorem) Assume Ω is a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1. Suppose $1 \leq p < n$. Then,

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \leq q < p^*$.

Poincare's Inequality (Evans Sec. 5.8.1)

- **Theorem:** (Poincare's inequality) Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p, Ω, such that

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

for each function $u \in W^{1,p}(\Omega)$.

Difference Quotients (Evans Sec. 5.8.2)

- **Definition:** Assume $u : \Omega \to \mathbb{R}$ is in $L^1_{\text{loc}}(\Omega)$ and $V \subset\subset \Omega$. Then the i^{th}-difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} \qquad (i = 1, \ldots, n)$$

for $x \in V$ and $h \in \mathbb{R}$ with $0 < |h| < \text{dist}(V, \partial\Omega)$. We then define the difference quotient to be the vector

$$D^h u := \left(D_1^h u, \ldots, D_n^h u\right)$$

- **Theorem:** (Difference quotients and weak derivatives)

1. Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then for each $V \subset\subset \Omega$

$$\|D^h u\|_{L^p(V)} \leq C\|Du\|_{L^p(\Omega)}$$

for some constant C and all $0 < |h| < \frac{1}{2}\,\text{dist}(V, \partial\Omega)$.

2. Assume $1 < p < \infty$ and $u \in L^p(V)$. Then $u \in W^{1,p}(V)$ with $\|Du\|_{L^p(V)} \leq C$.

Sobolev Dual Space (Evans Sec. 5.9.1)

- **Definition:** We denote by $H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

- **Definition:** If $f \in H^{-1}(\Omega)$, we define the norm

$$\|f\|_{H^{-1}(\Omega)} := \sup\left\{\langle f, u\rangle : u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} \leq 1\right\}$$

- **Theorem:** (Characterization of H^{-1}) If $f \in H^{-1}(\Omega)$, then there exists $f^0, f^1, \ldots, f^n$ in $L^2(\Omega)$ such that

$$\langle f, v\rangle = \int_\Omega f^0 v + \sum_{i=1}^n f^i v_{x_i} dx \qquad \text{for } v \in H_0^1(\Omega)$$

and we identify $f \in H^{-1}(\Omega)$ with $f^0 - \sum_{i=1}^n f^i_{x_i}$

Elliptic Equations (Evans Sec. 6.1.1)

- **Definition:** Our focus is on the boundary-value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ and $u : \overline{\Omega} \to \mathbb{R}$ is the unknown. Here, $f : \Omega \to \mathbb{R}$ is given and L denotes a second order partial differential operator having either the form

$$Lu = -\sum_{i,j=1}^n \left(a^{ij(x)} u_{x_i}\right)_{x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x)u \qquad \text{(divergence form)}$$

or

$$Lu = -\sum_{i,j=1}^n a^{ij(x)} u_{x_i x_j} + \sum_{i=1}^n b^i(x) u_{x_i} + c(x)u \qquad \text{(nondivergence form)}$$

for given coefficient functions a^{ij}, b^i, c.

- **Definition:** We say a partial differential operator L is uniformly elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

Weak Solution (Evans Sec. 6.1.2)

- **Definition:** The bilinear form $B[\cdot,\cdot]$ associated with the divergence form elliptic operator L above is

$$B[u,v] := \int_\Omega \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \ dx$$

for $u, v \in H_0^1(\Omega)$.

- **Definition:** We say that $u \in H_0^1(\Omega)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if

$$B[u,v] = \langle f, v \rangle$$

for every $v \in H_0^1(\Omega)$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

- **Definition:** More generally, $u \in H_0^1(\Omega)$ is a weak solution of the boundary-value problem

$$\begin{cases} Lu = f^0 - \sum_{i=1}^n f^i_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

if

$$B[u,v] = \langle f, v \rangle$$

for all $v \in H_0^1(\Omega)$ where $\langle \cdot,\cdot \rangle$ denotes the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Lax Milgram Theorem (Evans Thm. 6.1.1)

- **Theorem:** Let H be a real Hilbert Space and assume that

$$B : H \times H \to \mathbb{R}$$

is a bilinear mapping, for which there exists constants $\alpha, \beta > 0$ such that

1. $|B[u,v]| \leq \alpha \|u\|_H \|v\|_H$ for $u, v \in H$.
2. $\beta \|u\|_H^2 \leq B[u,u]$ for $u \in H$.

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H (i.e. in the dual of H), then there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v \rangle$$

for all $v \in H$.

Regularity for Elliptic PDEs

We will assume that $\Omega \subset \mathbb{R}^n$ is bounded and open, $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where L has divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u$$

- **Theorem:** (Interior H^2-regularity) Assume

  $$a^{ij} \in C^1(\Omega) \;\; b^i, c \in L^\infty(\Omega) \qquad i,j = 1,\dots,n$$

  and $f \in L^2(\Omega)$. Then $u \in H^2_{\text{loc}}(\Omega)$ and for each open set $V \subset\subset \Omega$, we have the following estimate.

  $$\|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

- **Theorem:** (Higher interior regularity) Let m be a nonnegative integer and assume

  $$a^{ij}, b^i, c \in C^{m+1}(\Omega) \qquad i,j = 1,\dots,n$$

  and $f \in H^m(\Omega)$. Then, $u \in H^{m+2}_{\text{loc}}(\Omega)$ and for each $V \subset\subset \Omega$, we have the estimate

  $$\|u\|_{H^{m+2}(V)} \le C\left(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

- **Theorem:** (Infinite differentiability in the interior) Assume

  $$a^{ij}, b^i, c \in C^\infty(\Omega) \qquad i,j = 1\dots,n$$

  and $f \in C^\infty(\Omega)$. Then $u \in C^\infty$.

  We actually only needed $u \in H^1(\Omega)$ instead of $H_0^1(\Omega)$ in the above theorems.

- **Theorem:** (Boundary H^2-regularity) Assume

  $$a^{ij} \in C^1(\overline{\Omega}), \;\; b^i, c \in L^\infty(\Omega) \qquad i,j = 1,\dots,n$$

  Further assume $f \in L^2(\Omega)$ and $\partial\Omega$ is C^2. Then $u \in H^2(\Omega)$ and we have the estimate

  $$\|u\|_{H^2(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

- **Theorem:** (Higher boundary regularity) Let m be a nonnegative integer and assume

  $$a^{ij}, b^i, c \in C^{m+1}(\overline{\Omega}) \qquad i,j = 1,\dots,n$$

  Further assume $f \in H^m(\Omega)$ and $\partial\Omega$ is C^{m+2}. Then $u \in H^{m+2}(\Omega)$ and we have that estimate

  $$\|u\|_{H^{m+2}(\Omega)} \le C\left(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\right)$$

- **Theorem:** (Infinite differentiability up to the boundary) Assume

$$a^{ij}, b^i, c \in C^\infty(\overline{\Omega}) \qquad i, j = 1, \ldots, n$$

Further assume that $f \in C^\infty(\overline{\Omega})$ and $\partial\Omega$ is C^∞. Then $u \in C^\infty(\overline{\Omega})$.

Maximum Principle for Elliptic PDEs

- **Theorem:** (Weak maximum principle) Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \equiv 0$ in Ω.

  1. If $Lu \leq 0$ in Ω, then
  $$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$
  2. If $Lu \geq 0$ in Ω, then
  $$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$$

- **Lemma:** (Hopf's lemma) Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $c \equiv 0$ in Ω. Suppose further that $Lu \leq 0$ in Ω and there exists a point $x^0 \in \partial\Omega$ such that

$$u(x^0) > u(x) \qquad \text{for all } x \in \Omega$$

  Assume finally that Ω satisfies the interior ball condition at x^0; that is, there exists an open ball $B \subset \Omega$ with $x^0 \in \partial B$.

  Then,
$$\frac{\partial u}{\partial \nu}(x^0) > 0$$
  where ν is the outward unit normal to B at x^0. If $c \geq 0$ in Ω, then the same conclusion above holds, provided

$$u(x^0) \geq 0$$

- **Theorem:** (Strong maximum principle) Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \equiv 0$ in Ω. Suppose also that Ω is connected, open, and bounded. Then,

  1. If $Lu \leq 0$ in Ω and u attains its maximum over $\overline{\Omega}$ at an interior point, then u is constant within Ω.
  2. If $Lu \geq 0$ in Ω and u attains its minimum over $\overline{\Omega}$ at an interior point, then u is constant within Ω.

2 Part A

Brauer 1.7.2

Find all continuous nonnegative functions f on $0 \le t \le 1$ such that

$$f(t) \le \int_0^t f(s)ds$$

Proof. Notice that the condition above can be rewritten as

$$f(t) \le 0 + \int_0^t f(s)ds$$

Thus, by Gronwall's, $f(t) \le 0$, so only $f \equiv 0$ satisfies the condition. $\square$

Brauer 1.7.3

Let $f(t)$ be a nonnegative function satisfying

$$f(t) \le K_1 + \epsilon(t-\alpha) + K_2 \int_\alpha^t f(s)ds$$

on an interval $\alpha \le t \le \beta$, where ϵ, K_1, K_2 are given positive constants. Show that

$$f(t) \le K_1 e^{K_2(t-\alpha)} + \frac{\epsilon}{K_2}\left(e^{K_2(t-\alpha)} - 1\right)$$

Proof.

1. Let
$$U(t) = K_1 + \epsilon(t-\alpha) + K_2 \int_\alpha^t f(s)ds$$
so that $f(t) \le U(t)$.

2. Next, taking the derivative, we have
$$U'(t) = \epsilon + K_2 f(t) \le \epsilon + K_2 U(t)$$
$$U'(t) - K_2 U(t) - \epsilon \le 0$$
We'll force a product rule by multiplying by $e^{-K_2(t-\alpha)}$. Note that $-K_2(t-\alpha)$ and $-K_2 t$ have the same derivative. Thus, we have
$$e^{-K_2(t-\alpha)}U'(t) - K_2 e^{-K_2(t-\alpha)}U(t) - \epsilon e^{-K_2(t-\alpha)} \le 0$$
$$\frac{d}{dt}\left[U(t)e^{-K_2(t-\alpha)}\right] - \epsilon e^{-K_2(t-\alpha)} \le 0$$

3. Using FTC, we'll integrate over $[\alpha, t]$ to get

$$U(t)e^{-K_2(t-\alpha)} - U(\alpha) + \frac{\epsilon}{K_2}e^{-K_2(t-\alpha)} - \frac{\epsilon}{K_2} \leq 0$$

$$U(t)e^{-K_2(t-\alpha)} \leq U(\alpha) - \frac{\epsilon}{K_2}\left(e^{-K_2(t-\alpha)} - 1\right)$$

$$U(t) \leq K_1 e^{K_2(t-\alpha)} + \frac{\epsilon}{K_2}\left(e^{K_2(t-\alpha)} - 1\right)$$

and since $f(t) \leq U(t)$ by hypothesis, we are done.

□

Gronwall's Inequality Differential Form

Let v, u be continuous functions on the interval $\alpha \leq t \leq \beta$. If u is differentiable on (α, β) and satisfies

$$u'(t) \leq v(t)u(t) \qquad t \in (\alpha, \beta)$$

then

$$u(t) \leq u(\alpha) \exp\left\{\int_\alpha^t v(s)ds\right\}$$

Proof. Define

$$w(t) = \exp\left\{\int_\alpha^t v(s)ds\right\}$$

so that $w(t) > 0$ and $w(\alpha) = 1$. Next, observe that

$$w'(t) = w(t)v(t) \implies v(t) = \frac{w'(t)}{w(t)}$$

so by substitution,

$$u'(t) \leq u(t)v(t) \leq \frac{u(t)w'(t)}{w(t)}$$

$$w(t)u'(t) - u(t)w'(t) \leq 0$$

$$\frac{w(t)u'(t) - u(t)w'(t)}{[w(t)]^2} \leq 0 \qquad \text{(multiply by } 1/[w(t)]^2 \text{ since } w > 0)$$

$$\frac{d}{dt}\left(\frac{u(t)}{w(t)}\right) \leq 0 \qquad \text{(force quotient rule)}$$

Now integrate over $[\alpha, t]$ to get

$$\frac{u(t)}{w(t)} - \frac{u(\alpha)}{w(\alpha)} \leq 0$$

$$u(t) \leq u(\alpha)w(t) = u(\alpha)\exp\left\{\int_\alpha^t v(s)ds\right\}$$

□

Brauer 1.7.4

Find all continuous functions $f(t)$ such that

$$[f(t)]^2 = \int_0^t f(s)ds \qquad t \geq 0$$

Proof. We first notice that $f(0) = 0$. Next, let us consider the following cases

1. If $f(t_0) > 0$ for some $t_0 > 0$, then there exists an open ball $B_r(t_0)$ for which $f > 0$. Thus,
$$f(t) = \sqrt{[f(t)]^2} \qquad t \in B_r(t_0)$$
is differentiable on $B_r(t_0)$ so taking the derivative of our original equality,
$$\begin{aligned} 2f(t)f'(t) &= f(t) && (t \in B_r(t_0)) \\ 2f'(t) &= 1 && (f(t) > 0) \\ f(t) &= \frac{1}{2}t + c \end{aligned}$$
and $c = 0$ since $f(0) = 0$.

2. If $f(t_0) < 0$ for some $t_0 > 0$, then there exists an open ball $B_r(t_0)$ for which $f < 0$. Thus, by a similar process, we again have that
$$f(t) = \frac{1}{2}t$$
but since $f(0) = 0$, it is impossible to have $f < 0$ since $t \geq 0$ and our slope is positive.

Thus, since f is continous on $[0, \infty)$, we have only the case below:

$$f(t) = \begin{cases} 0 & t < a \\ \frac{1}{2}t & t \geq a \end{cases}$$

for $a \in [0, \infty]$. □

Brauer 2.1.2

Write the scalar linear equation $y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y' + a_n y = b$ as a system $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t)$

Proof. We first see that $y^{(n)}(t) = -a_1(t)y^{(n-1)}(t) - \cdots - a_{n-1}y'(t) - a_n(t)y + b(t)$. Now defining

$$y_1 = y, \;\; y_2 = y' = y_1', \;\; y_2 = y'' = y_2', \;\; \ldots, \;\; y_{n-1} = y^{(n-2)} = y_{n-2}', \;\; y_n = y^{(n-1)} = y_{n-1}'$$

Then we construct the system,

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
&\vdots \\
y_{n-1}' &= y_n \\
y_n' &= -a_1(t)y_{n-1} - \cdots - a_{n-1}(t)y_2 - a_n(t)y_1 + b(t)
\end{aligned}$$

Thus, in matrix notation, we have

$$\underbrace{\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_{n-1}' \\ y_n' \end{bmatrix}}_{\mathbf{y}'} = \underbrace{\begin{bmatrix} 0 & & & I_{n-1} & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \\ -a_n(t) & -a_{n-1}(t) & \cdots & -a_2(t) & a_1(t) & 0 \end{bmatrix}}_{\mathbf{A}(t)} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}}_{\mathbf{y}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{bmatrix}}_{\mathbf{g}(t)}$$

where I_{n-1} denotes the $(n-1)$-dimension identity matrix. □

Brauer 2.3.3

Suppose $A(t)$ and $g(t)$ are continuous for $-\infty < t < \infty$ and that

$$\int_{-\infty}^{\infty} |A(t)|dt < \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} |g(t)|dt < \infty$$

Show that the solution $\phi(t)$ of $y' = A(t)y + g(t)$ exists for $-\infty < t < \infty$ and compute a bound for $|\phi(t)|$ valid for $-\infty < t < \infty$.

Proof. Since A, g are continuous for all t and $F(t, y) := A(t)y + g(t)$ is continuous on

$$D = \{(t, y) : -\infty < t < \infty, -\infty < y < \infty\}$$

then by theorem 1.1, a unique continuous solution exists for $-\infty < t < \infty$ so long as $|\phi(t)| < \infty$ for all t.

To show ϕ is uniformly bounded, we first apply theorem 2.1 on a finite interval $-n \leq t \leq n$ on which a unique continuous solution $\phi(t)$ exists with $\phi(t_0) = \eta$, $|t_0| < n$, and $|\eta| < \infty$.

Since ϕ is a solution of the linear system, we have

$$\begin{aligned}
\int_{t_0}^{t} \phi'(s)ds &= \int_{t_0}^{t} A(s)\phi(s)ds + \int_{t_0}^{t} g(s)ds && (t_0 < t < n)\\
\phi(t) - \phi(t_0) &= \int_{t_0}^{t} A(s)\phi(s)ds + \int_{t_0}^{t} g(s)ds && \text{(FTC)}\\
|\phi(t)| &\leq |\eta| + \int_{t_0}^{t} |A(s)||\phi(s)|ds + \int_{t_0}^{t} |g(s)|ds && \text{(triangle ineq.)}\\
&\leq |\eta| + \int_{-\infty}^{\infty} |g(s)|ds + \int_{t_0}^{t} |A(s)||\phi(s)|ds && \text{(expand)}\\
|\phi(t)| &\leq \left(|\eta| + \int_{-\infty}^{\infty} |g(s)|ds\right) \exp\left\{\int_{t_0}^{t} |A(s)|ds\right\} && \text{(Gronwall)}\\
&\leq \left(|\eta| + \int_{-\infty}^{\infty} |g(s)|ds\right) \exp\left\{\int_{-\infty}^{\infty} |A(s)|ds\right\} && \text{(expand)}\\
&< \infty
\end{aligned}$$

Thus, ϕ is uniformly bounded for all $t \in (-\infty, \infty)$, so the solution may be extended to all $t \in (-\infty, \infty)$. □

Corollary of Brauer Thm. 2.2

A fundamental solution to the autonomous linear system, $X'(t) = AX$, is a nonsingular matrix-valued function, $\Phi : \mathbb{R} \to \mathbb{M}_{d\times d}$, with $\Phi'(t) = A\Phi(t)$.

(a) Show that $\Psi(t) = e^{At}$ is a fundamental solution satisfying $\Psi(0) = I_n$, the identity matrix.

(b) Show that $X(t) = \Phi(t)\Phi(0)^{-1}X_0$ is a solution to the IVP, $X'(t) = AX$, $X(0) = X_0$.

(c) Show that any fundmantal solution is of the form $\Phi(t) = e^{At}M$, for some non-singular matrix M.

Proof.

(a) First, we see that

$$\Psi(0) = e^{At}\Big|_{t=0} = \sum_{j=0}^{\infty} \frac{(At)^j}{j!}\Big|_{t=0} = I + At + \frac{A^2t^2}{2!} + \cdots\Big|_{t=0} = I$$

Next, we'll show that Ψ is a solution to the system.

$$\begin{aligned}\Psi'(t) &= \frac{d}{dt}\left[I + At + \frac{A^2t^2}{2!} + \cdots\right] \\ &= A + \frac{A^2t}{1!} + \frac{A^3t^2}{2!} + \cdots \\ &= A\left(\sum_{j=0}^{\infty} \frac{A^jt^j}{j!}\right) \\ &= A\Psi(t)\end{aligned}$$

Last, since $\Psi(0) = I_n$, then $\det \Psi(0) = 1$, so by Abel's formula, $\det \Psi(t) \geq 1$ for all t, so Ψ must be fundamental.

(b) It is clear that $X(0) = X_0$ and

$$X'(t) = \Phi'(t)\Phi(0)^{-1}X_0 = A\Phi(t)\Phi(0)^{-1}X_0 = AX(t)$$

(c) Let Φ be a fundamental solution of the above system. Then since $\Psi(t) = e^{At}$ is also a fundamental solution, then by definition, the columns of $\Psi(t)$ are linearly independent for each t and thus form a basis for the set of solutions of our system. Let $\Psi_j(t)$, $\Phi_j(t)$ denote the jth column of Ψ and Φ respectively. Then there exists constants $(c_{j,k})_{k=1}^n$ such that

$$\Phi_j(t) = \sum_{k=1}^{n} \Psi_k(t)c_{j,k} = \big(\Psi_1(t) \cdots \Psi_n(t)\big)\begin{pmatrix} c_{j,1} \\ c_{j,2} \\ \vdots \\ c_{j,n}\end{pmatrix} = \Psi(t)\begin{pmatrix} c_{j,1} \\ c_{j,2} \\ \vdots \\ c_{j,n}\end{pmatrix}$$

Thus,

$$\begin{aligned}\Phi(t) = (\Phi_1(t) \cdots \Phi_n(t)) &= \left(\sum_{k=1}^{n} \Psi_k(t)c_{1,k} \cdots \sum_{k=1}^{n} \Psi_k(t)c_{n,k}\right) \\ &= \left(\Psi(t)\begin{pmatrix} c_{1,1} \\ c_{1,2} \\ \vdots \\ c_{1,n}\end{pmatrix} \quad \cdots \quad \Psi(t)\begin{pmatrix} c_{j,1} \\ c_{j,2} \\ \vdots \\ c_{j,n}\end{pmatrix} \quad \cdots \quad \Psi(t)\begin{pmatrix} c_{n,1} \\ c_{n,2} \\ \vdots \\ c_{n,n}\end{pmatrix}\right) \\ &= \Psi(t)\underbrace{\begin{pmatrix} c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \cdots & c_{n,n}\end{pmatrix}}_{C}\end{aligned}$$

Now, to show that C is nonsingular, since Φ, Ψ are both fundamental solutions, then $\det \Phi(t) \neq 0$, and $\det \Psi(t) \neq 0$ for all t, so

$$\det(C) = \det(\Psi(0)^{-1}\Phi(0)) = \det(I_n\Phi(0)) = \det \Phi(0) \neq 0$$

□

Brauer 2.7.3

Show that if all eigenvalues have real part negative or zero, if those eigenvalues with zero real part are simple, and if $\int_{t_0}^{\infty} |g(s)|ds < \infty$, then every solution $\phi(t)$ of

$$y' = Ay + g(t) \qquad y(t_0) = \eta$$

on $0 \leq t_0 \leq t < \infty$ is bounded.

Proof. Since A is a constant matrix, then we know by variation of parameters, that the unique solution ϕ is

$$\phi(t) = e^{A(t-t_0)}\eta + e^{At}\int_{t_0}^{t} e^{-As}g(s)ds$$

Thus,

$$|\phi(t)| \leq |\eta e^{-At_0}| \cdot |e^{At}| + |e^{-At_0}| \cdot |e^{At}| \int_{t_0}^{\infty} |g(s)|ds$$

and by theorem 2.10, since $0 \geq \Re\{\lambda_k\}$ for $k = 1, \ldots, n$ where λ_k are the eigenvalues of A (λ_k not necessarily distinct), then there exists a constant $K > 0$ with

$$|e^{At}| \leq Ke^{0t} = K$$

Thus,

$$|\phi(t)| \leq K|\eta e^{-At_0}| \left(1 + \int_{t_0}^{\infty} |g(s)|ds\right) < M < \infty$$

for some $M > 0$, so $\|\phi\|_{L^\infty([t_0,\infty))} < \infty$. □

Brauer 3.1.2

Prove that the initial value problem

$$y'' + g(t, y(t)) = 0, \qquad y(0) = y_0, \quad y'(0) = z_0$$

where g is continuous in some region D containing $(0, y_0)$ is equivalent to the integral equation

$$y(t) = y_0 + z_0 t - \int_0^t (t-s)g(s, y(s))ds$$

Proof. We first see that the latter implies the former since

$$\begin{aligned}
y''(t) &= -\frac{d^2}{dt^2}\int_0^t (t-s)g(s,y(s))ds \\
&= -\frac{d}{dt}\left(\frac{d}{dt}\left[t\int_0^t g(s,y(s))ds - \int_0^t sg(s,y(s))ds\right]\right) \\
&= -\frac{d}{dt}\left(\int_0^t g(s,y(s))ds + tg(t,y(t)) - tg(t,y(t))\right) && \text{(FTC)} \\
&= -g(t,y(t))
\end{aligned}$$

To show that the former implies the latter, we first integrate our IVP.

$$\begin{aligned}
\int_0^s y''(\tau) + g(\tau,y(\tau))d\tau &= y'(s) - y'(0) + \int_0^s g(\tau,y(\tau))d\tau \\
&= y'(s) - z_0 + \int_0^s g(\tau,y(\tau))d\tau
\end{aligned}$$

Then, we integrate again,

$$\begin{aligned}
\int_0^t y'(s) - z_0 + \int_0^s g(\tau,y(\tau))d\tau ds &= y(t) - y(0) - z_0 t + \int_0^t\int_0^s g(\tau,y(\tau))d\tau ds \\
&= y(t) - y_0 - z_0 t + \int_0^t\int_0^s g(\tau,y(\tau))d\tau ds && (*)
\end{aligned}$$

Now using integration by parts on the outer integral (and choosing our u to be the inner integral, $v = 1$), we have

$$\begin{aligned}
\int_0^t \left(\int_0^s g(\tau,y(\tau))d\tau\right) ds &= s\int_0^s g(\tau,y(\tau))d\tau\Big|_{s=0}^{s=t} - \int_0^t sg(s,y(s))ds \\
&= t\int_0^t g(\tau,y(\tau))d\tau - \int_0^t sg(s,y(s))ds \\
&= \int_0^t (t-s)g(s,y(s))ds && \text{(relabeling)}
\end{aligned}$$

Plugging the above into $(*)$ gives the desired result. □

Brauer 3.1.13

Consider the integral equation

$$y(t) = e^{it} + \alpha\int_t^\infty \sin(t-s)\frac{y(s)}{s^2}ds \qquad \alpha \in \mathbb{C}$$

Define the successive approximations

$$\begin{cases} \phi_0(t) \equiv 0 \\ \phi_n(t) = e^{it} + \alpha \int_t^\infty \sin(t-s)\frac{\phi_{n-1}(s)}{s^2} ds \end{cases}$$

(a) Show by induction that

$$|\phi_n(t) - \phi_{n-1}(t)| \le \frac{|\alpha|^{n-1}}{(n-1)!t^{n-1}} \qquad t \in [1,\infty), n \in \mathbb{N}$$

(b) Show that the ϕ_n converges uniformly on $[1,\infty)$ to a continuous function ϕ.

(c) Show that the limit ϕ satisfies the above integral equation.

(d) Show that the limit ϕ satisfies

$$|\phi(t)| \le e^{|\alpha|}$$

Proof. (a) For $n = 1$, we see that

$$|\phi_1(t) - \phi_0(t)| = |\phi_1(t)| = \left| e^{it} + \alpha \int_t^\infty \sin(t-s)\frac{\phi_0(s)}{s^2} ds \right| = |e^{it}| = 1 = \frac{|\alpha|^{1-1}}{(1-1)!t^{1-1}}$$

Assuming the result holds for n, then for $n+1$, we have

$$\begin{aligned} |\phi_{n+1}(t) - \phi_n(t)| &\le |\alpha| \int_t^\infty \frac{|\phi_n(s) - \phi_{n-1}(s)|}{s^2} ds \\ &\le |\alpha| \int_t^\infty \frac{|\alpha|^{n-1}}{(n-1)!s^{n+1}} ds && \text{(inductive hypothesis)} \\ &= \frac{|\alpha|^n}{(n-1)!} \int_t^\infty s^{-n-1} ds \\ &= \frac{|\alpha|^n}{n!t^n} \end{aligned}$$

(b) Let $\epsilon > 0$ and consider $n, m, N \in \mathbb{N}$ with $n \geq m \geq N$.

$$\begin{aligned}
|\phi_n(t) - \phi_m(t)| &\leq \sum_{k=0}^{n-m-1} |\phi_{n-k}(t) - \phi_{n-1-k}(t)| \\
&\leq \sum_{k=0}^{n-m-1} \frac{|\alpha|^{n-1-k}}{(n-1-k)!t^{n-1-k}} \\
&\leq \sum_{k=0}^{n-m-1} \frac{|\alpha|^{n-1-k}}{(n-1-k)!} && \text{(since } t \geq 1\text{)} \\
&\leq \sum_{k=0}^{n-N-1} \frac{|\alpha|^{n-1-k}}{(n-1-k)!} \\
&< \sum_{k=0}^{n-N-1} \frac{1}{\sqrt{2\pi(n-1-k)}} \left(\frac{|\alpha|e}{n-1-k}\right)^{n-1-k} && \text{(Stirling's approx.)} \\
&< \sum_{k=0}^{n-N-1} \left(\frac{|\alpha|e}{n-1-k}\right)^{n-1-k}
\end{aligned}$$

Thus, choosing $N > \frac{|\alpha|e}{\epsilon}$, we have

$$|\phi_n(t) - \phi_m(t)| < \sum_{k=N}^{n-1} \epsilon^k < \sum_{k=N}^{\infty} \epsilon^k = \frac{\epsilon^N}{1-\epsilon} < \epsilon$$

Thus, $(\phi_n)_{n=1}^{\infty}$ is uniformly Cauchy, and hence converges uniformly by Cauchy's criterion to some ϕ. Moreover, since ϕ_n is continuous for all n, then ϕ must also be continuous.

(c) To show ϕ satisfies the given integral equation, observe

$$\begin{aligned}
e^{it} + \alpha \int_t^{\infty} \sin(t-s)\frac{\phi(s)}{s^2} ds &= e^{it} + \alpha \int_t^{\infty} \sin(t-s) \lim_{n\to\infty} \frac{\phi_n(s)}{s^2} ds \\
&= \lim_{n\to\infty} \left(e^{it} + \alpha \int_t^{\infty} \sin(t-s)\frac{\phi_n(s)}{s^2} ds \right) && \text{(unif. conv.)} \\
&= \lim_{n\to\infty} \phi_{n+1}(t) \\
&= \phi(t)
\end{aligned}$$

(d) Observe that

$$
\begin{aligned}
|\phi_n(t)| &= \left|\sum_{k=1}^{n} \phi_k(t) - \phi_{k-1}(t)\right| \\
&\leq \sum_{k=1}^{n} |\phi_k(t) - \phi_{k-1}(t)| \\
&\leq \sum_{k=1}^{n} \frac{|\alpha|^{k-1}}{(k-1)!t^{k-1}} \\
&< \sum_{k=0}^{\infty} \frac{\left(\frac{|\alpha|}{t}\right)^k}{k!} \\
&= e^{\frac{|\alpha|}{t}} \\
&\leq e^{|\alpha|}
\end{aligned}
$$

□

Tonelli Iteration Scheme

Fix $T > 0, n \in \mathbb{N}$ and define the *Tonelli sequence* by

$$
x_n(t) = \begin{cases} x_0 & 0 \leq t \leq \frac{T}{n} \\ x_0 + \displaystyle\int_0^{t-\frac{T}{n}} f(s, x_n(s))ds & \frac{T}{n} \leq t \leq T \end{cases}
$$

for the initial value problem

$$
x'(t) = f(t, x(t)) \qquad x(0) = x_0
$$

Using this iteration scheme as an alternative to the successive approximations, state the proper existence theorem and prove it.

Solution: *Theorem:* Suppose f and $\partial f/\partial x$ are continuous on the closed rectangle

$$
R = [-a, a] \times [x_0 - b, x_0 + b]
$$

Then the Tonelli sequence converges uniformly on the interval

$$
I = [0, c] \qquad c = \min\left\{a, T, \frac{b}{\|f\|_\infty}\right\}
$$

to a solution of the initial value problem given above.

Proof. We'll first prove that x_k is well-defined for all $k \in \mathbb{N}$. If $c \leq \frac{T}{k}$, then $x_k \equiv x_0$ for all $t \in [0, c]$ and it is clear that $(t, x_0) \in R$ for $t \in [0, c]$. Now, if $c > \frac{T}{k}$ and x_k fails to be defined on $[0, c]$, then there exists some $t' \in \left(\frac{T}{k}, c\right]$ such that $x_k(t') \notin [x_0 - b, x_0 + b]$, so $|x_k(t') - x_0| > b$. However, observe that

$$\begin{aligned}
|x_k(t') - x_0| &= \left| \int_0^{t' - \frac{T}{k}} f(s, x_k(s)) ds \right| \\
&\leq \int_0^{t' - \frac{T}{k}} |f(s, x_k(s))| ds \\
&\leq \|f\|_\infty \left(t' - \frac{T}{k} \right) \\
&\leq \|f\|_\infty \left(c - \frac{T}{k} \right) \\
&\leq b - \frac{\|f\|_\infty T}{k} \\
&< b
\end{aligned}$$

a contradiction. Thus, x_k is well-defined for all $t \in [0, c]$ for every $k \in \mathbb{N}$

Next, we will show that x_k is continuous on $[0, c]$. Indeed, if $t_1, t_2 \in \left[\frac{T}{k}, c\right]$ with $t_1 < t_2$, then

$$|x_k(t_1) - x_k(t_2)| \leq \int_{t_1 - \frac{T}{k}}^{t_2 - \frac{T}{k}} |f(s, x_k(s))| ds \leq \|f\|_\infty |t_2 - t_1|$$

thus showing that x_k is continuous on $\left[\frac{T}{k}, c\right]$. It is clear that the same estimate holds for all $t_1, t_2 \in [0, c]$, so x_k is continuous on $[0, c]$ for every $k \in \mathbb{N}$.

Now, let $\epsilon > 0$ and let $n > m \geq N$ all be natural numbers with $\frac{T}{N} < c$. Since $f, \partial f / \partial x$ are continuous on R compact, then we know that f is Lipschitz and bounded on R. Now let us observe the following case:

For $t \in [0, c]$, if $t \geq \frac{T}{m}$, then we have that

$$\begin{aligned}
|x_n(t) - x_m(t)| &= \left| \int_0^{t - \frac{T}{n}} f(s, x_n(s)) ds - \int_0^{t - \frac{T}{m}} f(s, x_m(s)) ds \right| \\
&\leq \left| \int_{t - \frac{T}{m}}^{t - \frac{T}{n}} f(s, x_n(s)) ds \right| + \left| \int_0^{t - \frac{T}{m}} f(s, x_n(s)) - f(s, x_m(s)) ds \right| \\
&\leq \int_{t - \frac{T}{m}}^{t - \frac{T}{n}} |f(s, x_n(s))| ds + \int_0^{t - \frac{T}{m}} |f(s, x_n(s)) - f(s, x_m(s))| ds \\
&\leq \|f\|_\infty \left(\frac{T}{m} - \frac{T}{n} \right) + \int_0^{t - \frac{T}{m}} D |x_n(s) - x_m(s)| ds \qquad \text{(Lipschitz)}
\end{aligned}$$

where D is the Lipschitz constant of f. Next, since $|(x_n - x_m)(t)|$ is clearly nonnegative and x_n is continuous for all n, then we may apply the Gronwall inequality to get

$$
\begin{aligned}
|x_n(t) - x_m(t)| &\leq \|f\|_\infty \left(\frac{T}{m} - \frac{T}{n}\right) \exp\left\{\int_0^{t-\frac{T}{m}} D ds\right\} \\
&= \|f\|_\infty \left(\frac{T}{m} - \frac{T}{n}\right) e^{D\left(t-\frac{T}{m}\right)} \\
&< \|f\|_\infty \frac{T}{m} e^{Dc}
\end{aligned}
$$

Thus, if we further suppose $N > \frac{\|f\|_\infty T e^{Dc}}{\epsilon}$, then for $n, m \geq N$, we have

$$
|x_n(t) - x_m(t)| < \frac{\|f\|_\infty T e^{Dc}}{N} < \epsilon
$$

We'll now show that this choice of N also holds to show that (x_n) is Cauchy for all $t \in [0, c]$.

Indeed, if $t < \frac{T}{n}$, then (x_n) is clearly Cauchy. If $t \in \left[\frac{T}{n}, \frac{T}{m}\right]$, then

$$
\begin{aligned}
|x_n(t) - x_m(t)| &= \left|\int_0^{t-\frac{T}{n}} f(s, x_n(s)) ds\right| \\
&\leq \|f\|_\infty \left(t - \frac{T}{n}\right) \\
&\leq \|f\|_\infty \left(\frac{T}{m} - \frac{T}{n}\right) \\
&< \|f\|_\infty \frac{T}{m}
\end{aligned}
$$

Thus, (x_n) is uniformly Cauchy, so it must converge uniformly to some function x. To show that x satisfies the integral equation

$$
x(t) = x_0 + \int_0^t f(s, x(s)) ds
$$

we see that

$$
x_n(t) = x_0 + \int_0^t f(s, x_n(s)) ds - \int_{t-\frac{T}{n}}^t f(s, x_n(s)) ds
$$

and since

$$
\lim_{n \to \infty} \left|\int_{t-\frac{T}{n}}^t f(s, x_n(s)) ds\right| \leq \lim_{n \to \infty} \|f\|_\infty \frac{c}{n} = 0
$$

we must have that

$$\lim_{n\to\infty} x_n(t) = x_0 + \lim_{n\to\infty}\int_0^t f(s, x_n(s))ds$$
$$x(t) = x_0 + \int_0^t f(s, x(s))ds \qquad (f \text{ continuous})$$

Next, if (t_n) is a convergent sequence to t, then

$$|x(t_n) - x(t)| \le |x(t_n) - x_n(t_n)| + |x_n(t_n) - x_n(t)| + |x_n(t) - x(t)|$$

and each of the three terms above can be made arbitrarily small by continuity of x_n and uniform convergence of x_n to x, so x is continuous on $[0, c]$. Last, it is clear that $x(0) = x_0$ since $(x_n(0))$ is the constant sequence (x_0).

□

Note that we can actually relax the condition that $\partial f/\partial x$ is bounded on R. Instead of using Lipschitz and Gronwall's to get our result, we need to employ Arzela-Ascoli.

Also, this theorem is sometimes referred to as the Cauchy-Peano (existence) theorem.

To remark about why we don't have an issue of circularity with the Tonelli sequence consider the following argument for why $x_n(t)$ is well-defined for all $t \in [0, T]$

$$\begin{cases} x_n(t) = x_0 & t \in [0, T/n] \\ x_n(t) = x_0 + \int_0^{t-T/n} f(s, x_0)ds =: y_1(t) & t \in [T/n, 2T/n] \\ x_n(t) = x_0 + \int_0^{t-T/n} f(s, x_n(s))ds = x_0 + \int_0^{t-T/n} f(s, y_1(s))ds =: y_2(t) & t \in [2T/n, 3T/n] \\ \vdots & \\ x_n(t) = x_0 + \int_0^{t-T/n} f(s, y_{k-1}(s))ds =: y_k(t) & t \in \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right] \\ \vdots & \end{cases}$$

At each stage of the above calculation, $x_n(t)$ is well-defined (since all terms involved are ultimately constants), so we can induct on k to show that $x_n(t)$ is well defined for all $t \in [0, T]$.

Strogatz 3.4.14

Consider the system $x' = rx + x^3 - x^5$, which exhibits a subcritical pitchfork bifurcation.

1. Find algebraic expressions for all the fixed points as r varies.

2. Sketch the vector field as r varies. Be sure to indicate all the fixed points and their stability.

3. Calculate r_s, the parameter at which the nonzero fixed points are born in a saddle-node bifurcation.

Solution: Setting $x' = 0$, we see that $rx + x^3 - x^5 = x(r + x^2 - x^4)$, so the second term is quadratic in x^2 and $x^* = 0$ is always a fixed point.

$$x^2 = \frac{-1 \pm \sqrt{1+4r}}{-2}$$
$$x = \pm\sqrt{\frac{-1 \pm \sqrt{1+4r}}{-2}}$$

Now, let us consider some cases:

(1) For $r < -\frac{1}{4}$, the discriminant will be negative, producing no additional fixed points.

(2) At $r = -\frac{1}{4}$, the discriminant is zero, so we gain two additional fixed points, $\pm\sqrt{\frac{1}{2}}$.

(3) For $r \in \left(-\frac{1}{4}, 0\right)$, no imaginary terms arise, so we gain 4 additional fixed points.

(4) For $r = 0$, $-1 + \sqrt{1+4r} = 0$, so we have only have 2 additional fixed points since this zero merges back with the existing $x^* = 0$.

(5) Last, for $r > 0$, we have the 2 fixed points from the previous case.

We note that $r_s = -\frac{1}{4}$ since at that parameter and two fixed points are born, at $\pm\sqrt{\frac{1}{2}}$. As r increases past r_s, each of these fixed points then split into pairs of fixed points.

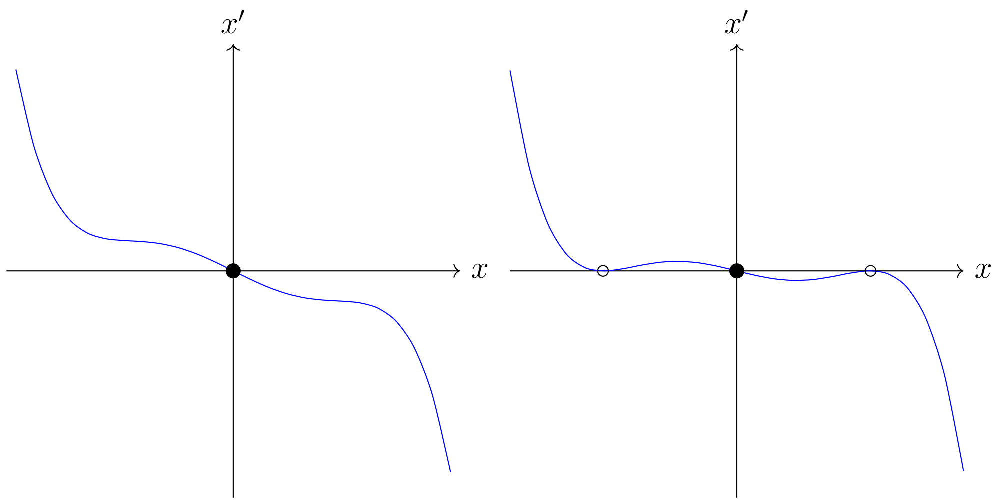

Figure 1: Left: $r < -\frac{1}{4}$, Right: $r = -\frac{1}{4}$

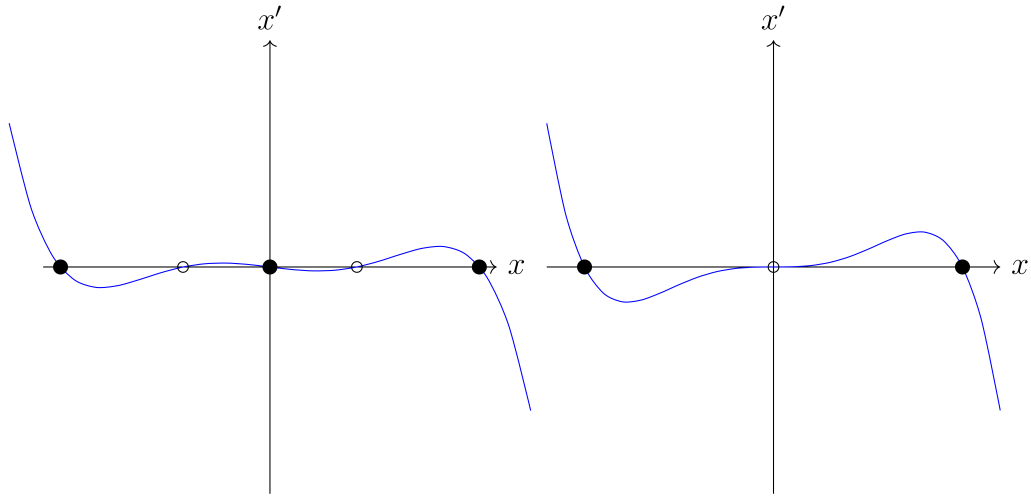

Figure 2: Left: $r \in \left(-\frac{1}{4}, 0\right)$, Right: $r \geq 0$

Strogatz 3.4.10

For the system below, find the values of r at which bifurcations occur and classify those. Finally, sketch the bifurcation diagram of fixed points r vs x^*.

$$x' = rx + \frac{x^3}{1+x^2}$$

Solution: Solving $x' = 0$, we have

$$x\left((r+1)x^2 + r\right) = 0$$

So we have a constant fixed point $x^* = 0$. Examining the other term, we have

$$x^{*2} = \frac{-r}{r+1} \qquad r \neq -1$$

In order to have fixed points, we require the right side to be nonnegative, so let us consider cases for r:

1. If $r > -1$, then $r + 1 > 0$, so for $\frac{-r}{r+1} \geq 0$, we have $r \leq 0$. Thus, the valid interval which produces fixed points is $r \in (-1, 0]$ with fixed points

$$x^* = \pm\sqrt{\frac{-r}{r+1}}$$

2. If $r < -1$, then $-r > 0$ and $r + 1 < 0$, so their quotient is negative so no additional fixed points come from this case.

Using the above information about the fixed points, we see that at $r_p = 0$, represents a pitchfork bifurcation since the split that happens occurs to an existing bifurcation point. In order to see which pitchfork bifurcation occurs, we will check the stability of $x^* = 0$ for values of $r > 0$. Starting with the left of $x^* = 0$, for $r > 0$, we have

$$x'\Big|_{x<0} = rx + \frac{x^3}{1+x^2}\Big|_{x<0} < 0$$

so $x^* = 0$ must be unstable since points on the left are moving away from it until $r = -1$, at which the two branches disappear. Thus, we must have a *subcritical pitchfork* since $x^* = 0$ will switch from unstable to stable at $r_p = 0$

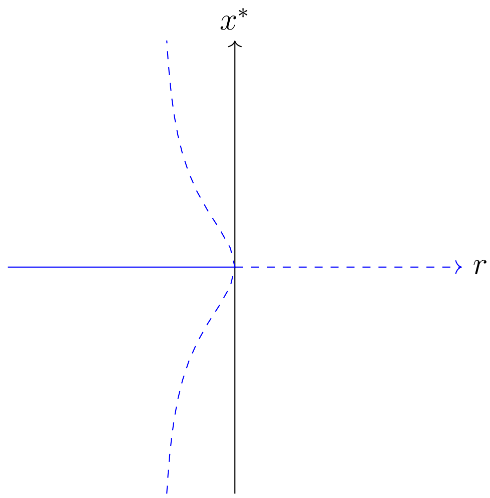

Figure 3: Bifurcation Diagram

3 Part B

Evans 2.5.1

Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \mathbb{R}^n \times (0, \infty) \\ u = g & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Solution: Given the observation

$$\frac{\partial}{\partial t}[e^{ct}u] = e^{ct}(cu + u_t)$$

we multiply our IVP by e^{ct} and letting $v = e^{ct}u$, we have

$$\begin{cases} v_t + b \cdot Dv = 0 & \mathbb{R}^n \times (0, \infty) \\ v = e^{ct}g & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Thus, using our solution to the transport problem, we have that

$$v(x,t) = g(x - tb) \Leftrightarrow \boxed{u(x,t) = e^{-ct}g(x - tb)}$$

Evans 2.5.2

Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if $O \in \mathbb{M}_{n\times n}$ is an orthogonal matrix and we define

$$v(x) := u(Ox)$$

then $\Delta v = 0$.

Proof. Let $O = (a_{ij})_{i,j=1}^n$. Then

$$Ox = \left(\sum_{i=1}^n a_{ji}x_i \right)_{j=1}^n$$

so we'll denote $y_j = \sum_{i=1}^n a_{ji}x_i$ so that u has the form

$$u = u(y_1(x_1, \dots, x_n), \dots, y_n(x_1, \dots, x_n))$$

Then taking the partial w.r.t. x_k, we use the total derivative:

$$\begin{aligned}
\frac{\partial v}{\partial x_k} &= \sum_{j=1}^{n} \frac{\partial u}{\partial y_j}\frac{\partial y_j}{\partial x_k} \\
&= \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} a_{jk} \\
\frac{\partial^2 v}{\partial {x_k}^2} &= \frac{\partial}{\partial x_k}\sum_{j=1}^{n} \frac{\partial u}{\partial y_j} a_{jk} \\
&= \sum_{j=1}^{n} a_{jk} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial y_j \partial y_i}\frac{\partial y_i}{\partial x_k} \\
&= \sum_{j=1}^{n} a_{jk} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial y_j \partial y_i} a_{ik} \\
\Delta v = \sum_{k=1}^{n} \frac{\partial^2 v}{\partial x_k^2} &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_{jk} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial y_j \partial y_i} a_{ik} \\
&= \sum_{j,i=1}^{n} \frac{\partial^2 u}{\partial y_j \partial y_i} \sum_{k=1}^{n} a_{jk}(a_{ki})^T
\end{aligned}$$

By orthogonality, we know that $\sum_{k=1}^{n} a_{jk}(a_{ki})^T = 1$ iff $j = k$ and it is zero otherwise. Thus,

$$\Delta v = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial {y_j}^2} = \Delta u = 0$$

□

Note that polar coordinates are defined by $x \mapsto ry$ where $r = |x|$ and $y \in \partial B_1(0)$

Mean Value Theorem for Laplace's equation

If $u \in C^2(\Omega)$ is harmonic, then

$$u(x) = ⨍_{\partial B_r(x)} u(y)dS(y) = ⨍_{B_r(x)} u(y)dy$$

Proof. Begin by defining

$$\begin{aligned}
\phi(r) :&= \fint_{\partial B_r(x)} u(y)dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_r(x)} u(y)dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B_1(0)} u(x+rz)r^{n-1}dS(z) && \text{(Change of variables (Polar))} \\
&= \frac{1}{n\alpha(n)} \int_{\partial B_1(0)} u(x+rz)dS(z) \\
&= \fint_{\partial B_1(0)} u(x+rz)dS(z)
\end{aligned}$$

Next, taking the derivative with respect to r,

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B_1(0)} Du(x+rz)zdS(z) \\
&= \fint_{\partial B_r(x)} Du(y)\frac{y-x}{r}dS(y) && \text{(change variables back to original)} \\
&= \fint_{\partial B_r(x)} \frac{\partial u}{\partial \nu}dS(y) && (Du(y)\tfrac{y-x}{r} \text{ is the unit normal)} \\
&= \frac{r}{n}\fint_{B_r(x)} \Delta u(y)dy && \text{(Gauss-Green Theorem)} \\
&= 0
\end{aligned}$$

Thus, ϕ is constant in r, so

$$\int_{\partial B_r(x)} u(y)dS(y) = \phi(r) = \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\fint_{\partial B_t(x)} u(y)dS(y) = u(x)$$

hence showing the result over a sphere. To show the result over the ball, we use polar coordinates,

$$\begin{aligned}
\int_{B_r(x)} u(y)dy &= \int_0^r \left(\int_{\partial B_t(x)} u(y)dS(y)\right) dt \\
&= \int_0^r \left(n\alpha(n)t^{n-1}\fint_{\partial B_t(x)} u(y)dS(y)\right) dt \\
&= \int_0^r n\alpha(n)t^{n-1}u(x)dt && \text{(mean value formula over the sphere)} \\
&= \alpha(n)r^n u(x)
\end{aligned}$$

Thus, dividing $\alpha(n)r^n$ to the other side, we have

$$\fint_{B_r(x)} u(y)dy = u(x)$$

□

Evans 2.5.3

Modify the proof of the mean-value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B_r(0)} g(x)dS(x) + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x)dx$$

provided

$$\begin{cases} -\Delta u = f & B_r(0) \\ u = g & \partial B_r(0) \end{cases}$$

Method 1

Proof. From the proof of the mean value formula, we know that if we define $\phi(r) := \fint_{\partial B_r(0)} u(y)dS(y)$, then

$$\phi'(r) = \frac{r}{n} \fint_{B_r(0)} \Delta u(y)dy$$

The trick now is to use the fundamental theorem of calculus in r to get us the $u(0)$ and $\phi(r)$ terms.

$$\begin{aligned} \phi(r) - \phi(\epsilon) &= \int_\epsilon^r \phi'(t)dt, \qquad \text{for } 0 < \epsilon < r \\ &= \int_\epsilon^r \frac{t}{n} \frac{1}{\alpha(n)t^n} \left(\int_{B_t(0)} \Delta u(y)dy \right) dt \\ &= \frac{1}{n\alpha(n)} \int_\epsilon^r t^{1-n} \left(\int_{B_t(0)} \Delta u(y)dy \right) dt \end{aligned}$$

To get the rest of the terms, we'll use integration by parts on the outermost integral. Continuing the equality from above, we have

$$\begin{aligned} &= \frac{1}{n\alpha(n)} \left[-\int_\epsilon^r \frac{t^{2-n}}{2-n} \left(\frac{d}{dt} \int_{B_t(0)} \Delta u(y)dy \right) dt + \left(\frac{1}{2-n} t^{2-n} \int_{B_t(0)} \Delta u(y)dy \right|_{t=\epsilon}^{t=r} \right] \\ &= \frac{1}{n(2-n)\alpha(n)} \int_\epsilon^r t^{2-n} \int_{\partial B_t(0)} f(y)dS(y)dt + \frac{1}{n(2-n)\alpha(n)} r^{2-n} \int_{B_r(0)} \Delta u(y)dy \\ &\quad - \frac{1}{n(2-n)\alpha(n)} \epsilon^{2-n} \int_{B_\epsilon(0)} \Delta u(y)dy \\ &= \underbrace{\frac{1}{n(2-n)\alpha(n)} \int_\epsilon^r t^{2-n} \int_{\partial B_t(0)} f(y)dS(y)dt}_{H} + \underbrace{\frac{1}{n(n-2)\alpha(n)} r^{2-n} \int_{B_r(0)} f(y)dy}_{I} \\ &\quad + \underbrace{\frac{1}{n(2-n)\alpha(n)} \epsilon^{2-n} \int_{B_\epsilon(0)} f(y)dy}_{J} \end{aligned}$$

Considering each integral separately, we'll start with J.

$$
\begin{aligned}
J &= \frac{1}{n(2-n)\alpha(n)}\epsilon^{2-n}\int_{B_\epsilon(0)} f(y)dy \\
|J| &\leq \frac{1}{n(n-2)\alpha(n)}\epsilon^{2-n}\int_{B(0,\epsilon)} |f|dy \\
&\leq \|f\|_\infty \frac{1}{n(n-2)\alpha(n)}\epsilon^{2-n}\int_{B(0,\epsilon)} dy \\
&= \frac{\|f\|_\infty}{n(n-2)\alpha(n)}\epsilon^{2-n}\alpha(n)\epsilon^n \\
&= \frac{\|f\|_\infty}{n(n-2)}\epsilon^2 \to 0 \qquad \text{as } \epsilon \to 0.
\end{aligned}
$$

Next, we see that I is already in the desired form, so we'll move onto H.

$$
\begin{aligned}
H &= \frac{1}{n(2-n)\alpha(n)}\int_\epsilon^r t^{2-n}\int_{\partial B_t(0)} f(y)dS(y)dt \\
&= \frac{1}{n(2-n)\alpha(n)}\int_\epsilon^r \int_{\partial B_t(0)} \frac{f(y)}{t^{n-2}}dS(y)dt \\
&= \frac{1}{n(2-n)\alpha(n)}\int_\epsilon^r \int_{\partial B_t(0)} \frac{f(y)}{t^{n-2}}dS(y)dt \\
&= \frac{1}{n(2-n)\alpha(n)}\int_0^r \int_{\partial B_t(0)} \frac{f(y)}{t^{n-2}}dS(y)dt - \frac{1}{n(2-n)\alpha(n)}\int_0^\epsilon \int_{\partial B_t(0)} \frac{f(y)}{t^{n-2}}dS(y)dt \\
&= \frac{1}{n(2-n)\alpha(n)}\int_{B_r(0)} \frac{f(y)}{|y|^{n-2}}dy - \underbrace{\frac{1}{n(2-n)\alpha(n)}\int_0^\epsilon \int_{\partial B_t(0)} \frac{f(y)}{t^{n-2}}dS(y)dt}_{K}
\end{aligned}
$$

Note above that $y \in \partial B_r(0)$ we have $|y| = r$. Next, we'll look at K.

$$
\begin{aligned}
|K| &\leq \frac{\|f\|_\infty}{n(n-2)\alpha(n)}\int_0^\epsilon \int_{\partial B_t(0)} t^{2-n}dS(y)dt \\
&= \frac{\|f\|_\infty}{n(n-2)\alpha(n)}\int_0^\epsilon t^{2-n}\left(\int_{\partial B_t(0)} dS(y)\right)dt \\
&= \frac{\|f\|_\infty}{n(n-2)\alpha(n)}\int_0^\epsilon t^{2-n}\left(n\alpha(n)t^{n-1}\right)dt \\
&\leq \frac{\|f\|_\infty}{n-2}\epsilon^2 \to 0 \qquad \text{as } \epsilon \to 0
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\lim_{\epsilon\to 0}\phi(r) - \phi(\epsilon) &= \lim_{\epsilon\to 0}(H + I + J) \\
\phi(r) - u(0) &= \frac{1}{n(2-n)\alpha(n)}\int_{B_r(0)} \frac{f(y)}{|y|^{n-2}}dydt + \frac{1}{n(n-2)\alpha(n)}\int_{B_r(0)} \frac{1}{r^{n-2}}f(y)dy \\
u(0) &= ⨍_{\partial B_r(0)} g(y)dS(y) + \frac{1}{n(n-2)\alpha(n)}\int_{B_r(0)}\left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}}\right)f(y)dy
\end{aligned}
$$

□

Method 2

Proof. Using Poisson's formula for the ball, we have

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} dS(y) + \int_{B_r(0)} f(y)G(x,y)dy$$

Let us define

$$\tilde{x} := \frac{rx}{|x|^2} \qquad x \in \mathbb{R}^n \backslash \{0\}$$

Then we note that $\tilde{x}$ is the *point dual to* x if $x \in B_r(0)$, so

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \qquad x, y \in B_r(0), x \neq y$$

Thus,

$$\begin{aligned} u(x) &= \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} dS(y) + \int_{B_r(0)} f(y)\left(\Phi(y-x) - \Phi(|x|(y-\tilde{x}))\right) dy \\ &= \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} dS(y) \\ &\qquad + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} f(y) \left(\frac{1}{|y-x|^{n-2}} - \frac{1}{||x|(y-\tilde{x})|^{n-2}} \right) dy \end{aligned}$$

Our goal now is to evaluate $u(0)$, but we note that $\Phi(x)$ has a singularity at $x = 0$, so instead we must take the limit as $|x| \to 0$ (equivalent to $\lim_{x \to 0}$ since Φ is radially symmetric). Observe that

$$\begin{aligned} \lim_{|x| \to 0} \Big| |x|(y-\tilde{x}) \Big| &= \lim_{|x| \to 0} \Big| |x|y - |x|\tilde{x} \Big| \\ &= \lim_{|x| \to 0} \lim_{|x| \to 0} \Big| |x|y - |x|\tilde{x} \Big| \\ &= \lim_{|x| \to 0} \Big| 0 - \lim_{|x| \to 0} |x|\tilde{x} \Big| \\ &= \lim_{|x| \to 0} \lim_{|x| \to 0} \Big| |x|\tilde{x} \Big| \\ &= \lim_{|x| \to 0} \left| |x| \frac{rx}{|x|^2} \right| \\ &= r \end{aligned}$$

Thus,

$$\begin{aligned}
u(0) &= \frac{r}{n\alpha(n)} \int_{\partial B_r(0)} \frac{g(y)}{|y|^n} dS + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} f(y) \left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}} \right) dy \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B_r(0)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} f(y) \left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}} \right) dy \\
&= ⨍_{\partial B_r(0)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} f(y) \left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}} \right) dy
\end{aligned}$$

□

Evans 2.5.4

Give a direct proof that if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic within a bounded open set Ω, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

(Hint: Define $u_\epsilon = u + \epsilon|x|^2$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\overline{\Omega}$ at an interior point.)

Proof. Define $u_\epsilon := u + \epsilon|x|^2$ and suppose that there exists $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \Omega^\circ$ such that u_ϵ attains its max at x^0. Next, since u is harmonic, then

$$\Delta u_\epsilon = \Delta u + 2\epsilon n = 2\epsilon n > 0$$

However, we now define $f_j : \mathbb{R} \to \mathbb{R}$ by

$$f_j(x) = u_\epsilon(x_1^0, \ldots, x_{j-1}^0, x, x_{j+1}^0, \ldots, x_n^0)$$

so f_j attains its max at $x = x_j^0$. Hence we know that $f_j''(x_j^0) < 0$. Thus, taking the Laplacian at x_0,

$$\Delta u_\epsilon(x^0) = \sum_{j=1}^n \frac{\partial^2 u_\epsilon}{\partial x_j{}^2}(x^0) = \sum_{j=1}^n f_j''(x_j^0) < 0$$

which contradicts $\Delta u_\epsilon > 0$. Thus, no such x^0 may exist, so

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

We then see that

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u_\epsilon = \max_{\partial\Omega} u_\epsilon = \max_{\partial\Omega} u + \epsilon|x|^2$$

Taking $\epsilon \to 0$, we have

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u$$

and since $\partial\Omega \subset \Omega$, we know that

$$\max_{\partial\Omega} u \le \max_{\overline{\Omega}} u$$

□

Evans 2.5.5

We say $v \in C^2(\overline{\Omega})$ is *subharmonic* if

$$-\Delta v \le 0, \quad \text{in } \Omega.$$

(a) Prove for subharmonic v that

$$v(x) \le \fint_{B_r(x)} v(y)dy, \quad \text{for all } B_r(x) \subset \Omega.$$

(b) Prove that therefore $\max_{\overline{\Omega}} v = \max_{\partial\Omega} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic whenever u is harmonic.

(a) *Proof.* Define $\phi(r) := \fint_{\partial B(x,r)} v(y)dS(y)$. Then we know that $\phi'(r) = \frac{r}{n} \fint_{B_r(x)} \Delta v(y)dy$. Since $-\Delta v \le 0$, then $\phi'(r) \ge 0$ for all $r \in \mathbb{R}^+$, so ϕ is increasing in r. Thus

$$v(x) = \lim_{r\to 0} \phi(r) \le \phi(r) = \fint_{\partial B_r(x)} v(y)dS(y).$$

Extending to $B_r(x)$ by polar coordinates, we have

$$\begin{aligned} \int_{B(x,r)} v(y)dy = \int_0^r n\alpha(n)t^{n-1} \left(\fint_{\partial B_t(x)} v(y)dS(y) \right) dt &\ge \int_0^r n\alpha(n)t^{n-1} v(x)dt \\ &= n\alpha(n)v(x)\frac{r^n}{n} \\ &= \alpha(n)r^n v(x). \end{aligned}$$

Hence, $v(x) \le \fint_{B(x,r)} v(y)dy$. □

(b) *Proof.* Suppose there exists $x_0 \in \Omega$ such that $v(x_0) = M = \max_{\overline{\Omega}} v$. Then for $r < \operatorname{dist}(x_0, \partial\Omega)$,

$$M = v(x_0) \le \fint_{B(x,r)} v(y)dy$$

Hence, $v(y) = M$ for all $y \in B_r(x)$. Now, consider the set $A := v^{-1}(\{M\})$. We have just shown that A must be open. Next, since $\{M\}$ is closed and v is continuous, then $A = v^{-1}(\{M\})$ must be closed as well. Assuming Ω is connected, then A must either be $\varnothing$ or Ω, but we know that $A \ne \varnothing$, so we are done. □

(c) *Proof.* Observe,

$$
\begin{aligned}
\Delta v = \Delta(\phi(u)) &= \sum_{i=1}^{n} (\phi(u))_{x_i x_i} \\
&= \sum_{i=1}^{n} \phi''(u)(u_{x_i})^2 + \phi'(u) u_{x_i x_i} && \text{(chain rule)} \\
&= \sum_{i=1}^{n} \phi''(u)(u_{x_i})^2 + \phi'(u)\Delta u \\
&= \phi''(u) \sum_{i=1}^{n} (u_{x_i})^2 && \text{(since } \Delta u = 0\text{)} \\
&\geq 0 && (\phi \text{ convex} \implies \phi'' \geq 0)
\end{aligned}
$$

Thus, $-\Delta v \leq 0$. □

(d) *Proof.* Observe,

$$
\begin{aligned}
\Delta(|Du|^2) &= \sum_{j=1}^{n} \sum_{i=1}^{n} 2 \left(\frac{\partial^2 u}{\partial x_j \partial x_i} \right)^2 + 2 \frac{\partial u}{\partial x_i} \cdot \frac{\partial}{\partial x_i} \left(\frac{\partial^2 u}{\partial x_j^2} \right) \\
&= \sum_{i,j=1}^{n} 2 \left(\frac{\partial^2 u}{\partial x_j \partial x_i} \right)^2 + \sum_{i=1}^{n} 2 \frac{\partial u}{\partial x_i} \cdot \frac{\partial}{\partial x_i} (\Delta u) \\
&= \sum_{i,j=1}^{n} 2 \left(\frac{\partial^2 u}{\partial x_j \partial x_i} \right)^2 \\
&\geq 0
\end{aligned}
$$

Thus, $-\Delta(|Du|^2) \leq 0$. □

Evans 2.5.6

Let Ω be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C depending only on Ω, such that

$$\max_{\Omega} |u| \leq C \left(\max_{\partial\Omega} |g| + \max_{\Omega} |f| \right)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Hint: Consider $-\Delta \left(u + \frac{|x|^2}{2n} \max_{\overline{\Omega}} |f| \right)$

Proof. Observe that

$$\begin{aligned}\Delta\left(u+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right) &= \Delta u + \max_{\overline{\Omega}}|f| \\ &= -f + \max_{\overline{\Omega}}|f| \qquad (x\in\Omega) \\ &\geq 0\end{aligned}$$

Thus, $-\Delta\left(u+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right)\leq 0$, so $\left(u+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right)$ is subharmonic. Thus, by Evans 2.5.5,

$$\begin{aligned}\max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}}\left(u+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right) &= \max_{\partial\Omega}\left(u+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right) \\ &\leq \max_{\partial\Omega} g + \left(\frac{1}{2n}\max_{\partial\Omega}|x|^2\right)\max_{\overline{\Omega}}|f| \\ &\leq C\left(\max_{\partial\Omega}|g| + \max_{\overline{\Omega}}|f|\right)\end{aligned}$$

Now, let $v := -u$ and we see that this produces an equivalent system

$$\begin{cases} -\Delta v = -f & \text{in } \Omega \\ v = -g & \text{on } \partial\Omega \end{cases}$$

Then, by a similar process as above, we have $\left(v+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right)$ is subharmonic, so

$$\begin{aligned}\max_{\overline{\Omega}}(v) \leq \max_{\overline{\Omega}}\left(v+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right) &= \max_{\partial\Omega}\left(v+\frac{|x|^2}{2n}\max_{\overline{\Omega}}|f|\right) \\ &\leq \max_{\partial\Omega}|-g| + \left(\frac{1}{2n}\max_{\partial\Omega}|x|^2\right)\max_{\overline{\Omega}}|f| \\ &\leq C\left(\max_{\partial\Omega}|g| + \max_{\overline{\Omega}}|f|\right)\end{aligned}$$

Thus,

$$\max_{\overline{\Omega}}(-u) \leq C\left(\max_{\partial\Omega}|g| + \max_{\overline{\Omega}}|f|\right)$$

but since $\max_{\overline{\Omega}}(-u) = -\min_{\overline{\Omega}} u$. Thus,

$$\min_{\overline{\Omega}} u \geq -C\left(\max_{\partial\Omega}|g| + \max_{\overline{\Omega}}|f|\right)$$

Thus, combining both results and then taking $\max_{\overline{\Omega}}$, we have

$$\max_{\overline{\Omega}}|u| \leq C\left(\max_{\partial\Omega}|g| + \max_{\overline{\Omega}}|f|\right)$$

□

Evans 2.5.7

Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is harmonic and positive in $B_r(0)$. This is an explicit form of Harnack's inequality.

Proof. Using Poisson's formula for the ball, $B_r(0)$, we have

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B_r(0)} \frac{g(y)}{|y-x|^n}dS(y) \qquad y \in \partial B_r(0)$$

Since $x \in B_r(0)$, then we know that

$$|y-x| \leq |r-x| \leq r+|x|$$

Thus,

$$\begin{aligned} u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B_r(0)} \frac{g(y)}{|y-x|^n}dS(y) &\geq \frac{r-|x|}{n\alpha(n)r}\int_{\partial B_r(0)} \frac{u(y)}{(r+|x|)^{n-1}}dS(y) \\ &= \frac{r-|x|}{n\alpha(n)r}\frac{1}{(r+|x|)^{n-1}}\int_{\partial B_r(0)} u(y)dS(y) \\ &= r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}⨍_{\partial B_r(0)} u(y)dS(y) \\ &= r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) && \text{(Mean Value)} \end{aligned}$$

Next, since $y \in \partial B_r(0)$

$$r = |y| \leq |y-x| + |x|$$

then $|y-x| \geq r-|x|$. Thus,

$$\begin{aligned} u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B_r(0)} \frac{g(y)}{|y-x|^n}dS(y) &\leq \frac{r+|x|}{n\alpha(n)r}\int_{\partial B_r(0)} \frac{u(y)}{(r-|x|)^{n-1}}dS(y) \\ &= \frac{r+|x|}{n\alpha(n)r}\frac{1}{(r-|x|)^{n-1}}\int_{\partial B_r(0)} u(y)dS(y) \\ &= r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}⨍_{\partial B_r(0)} u(y)dS(y) \\ &= r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0) && \text{(Mean Value)} \end{aligned}$$

□

Evans 2.5.8

Prove Poisson's formula for the ball. Assume $g \in C(\partial B_r(0))$ and define u by

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_r(0)} \frac{g(y)}{|y-x|^n} dS(y) \qquad x \in B_r(0)$$

Then,

(i) $u \in C^\infty(B_r(0))$.

(ii) $\Delta u = 0$ in $B_r(0)$.

(iii) $\lim_{\substack{x \to x_0 \\ x \in B_r(0)}} u(x) = g(x_0)$ for each $x_0 \in \partial B_r(0)$.

Hint: Since $u \equiv 1$ solves

$$\begin{cases} \Delta u = 0 & \text{in } B_r(0) \\ u = g & \text{on } \partial B_r(0) \end{cases}$$

for $g \equiv 1$, the theory automatically implies

$$\int_{\partial B_r(0)} K(x,y) dS(y) = 1 \qquad \text{where } K(x,y) = \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n}$$

for each $x \in B_r(0)$.

Vector Calculus Identities: Let $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{aligned} \nabla \cdot (\phi F) &= \phi(\nabla \cdot F) + (\nabla \phi) \cdot F \\ \Delta(\phi\psi) &= \phi\Delta\psi + 2(\nabla\phi) \cdot (\nabla\psi) + \psi\Delta\phi \end{aligned}$$

Note that we develop Poisson's formula for $u(x)$ as a solution to Laplace's equation under the assumption that a smooth solution exists. The theorem then shows that, indeed, $u(x)$ is smooth and it is a solution to Laplace's equation.

Proof. Let $u := r^2 - |x|^2$ and $v := |x-y|^{-n}$ so that

$$n\alpha(n)rK(x,y) = uv$$

Calculating, we have $\nabla u = -2x$, $\Delta u = -2n$ and

$$\begin{aligned}
\nabla v &= \nabla |x-y|^{-n} \\
&= -n|x-y|^{-(n+1)} \cdot \nabla|x-y| \\
&= -n|x-y|^{-(n+1)} \cdot \frac{x-y}{|x-y|} \\
&= -n\frac{x-y}{|x-y|^{n+2}} \\
\Delta v &= \nabla \cdot (\nabla v) \\
&= -n\left[|x-y|^{-(n+2)}n - (n+2)|x-y|^{-(n+3)}\frac{(x-y)}{|x-y|} \cdot (x-y)\right] \\
&= -n^2|x-y|^{-(n+2)} + n(n+2)\frac{|x-y|^2}{|x-y|^{n+4}} \\
&= \frac{-n^2}{|x-y|^{n+2}} + \frac{n^2+2n}{|x-y|^{n+2}} \\
&= \frac{2n}{|x-y|^{n+2}}
\end{aligned}$$

Then using the product rule for the Laplacian and noting that $|y| = r$,

$$\begin{aligned}
\Delta(uv) &= (r^2 - |x|^2)\frac{2n}{|x-y|^{n+2}} - 2n\frac{x-y}{|x-y|^{n+2}} \cdot (-2x) + |x-y|^{-n}(-2n) \\
|x-y|^{n+2}\Delta(uv) &= 2n|y|^2 - 2n|x|^2 + 4n|x|^2 - 4nx \cdot y - 2n|x-y|^2 \\
&= 2n\left(|y|^2 + |x|^2 - 2x\cdot y - |x|^2 - |y|^2 + 2x \cdot y\right) \\
&= 0
\end{aligned}$$

Thus, $\Delta K(x,y) = 0$, so K is harmonic. Moreover, since K is continuous for $x \neq y$, then

$$\Delta u(x) = \Delta\left(\int_{\partial B_r(0)} K(x,y)g(y)dS(y)\right) = \int_{\partial B_r(0)} \Delta K(x,y)g(y)dS(y) = 0$$

so u is harmonic and it is clear that $u \in C^2(B_r(0))$, so u satisfies the mean value property for all balls $B_s(x) \subseteq B_r(0)$, so by the smoothness theorem (Evans thm. 2.2.6), we have that $u \in C^\infty(B_r(0))$.

Next, note that when $g \equiv 1$, Then by the uniqueness of smooth solutions, $u \equiv 1$ solves,

$$\begin{cases} \Delta u = 0 & \text{in } B_r(0) \\ u = g & \text{on } \partial B_r(0) \end{cases}$$

and by Poisson's formula, if $x \in B_r(0)$,

$$1 = u(x) = \int_{\partial B_r(0)} K(x,y)g(y)dS(y) = \int_{\partial B_r(0)} K(x,y)dS(y)$$

Now let $\epsilon > 0$, $x_0 \in \partial B_r(0)$ and $x \in B_r(0)$. Since $g \in C(\partial B_r(0))$, we can choose $\delta > 0$ such that

$$|g(y) - g(x_0)| < \frac{\epsilon}{2} \qquad \text{when} \qquad |y - x_0| < \delta,\ \ y \in \partial B_r(0)$$

$$\begin{aligned}
|u(x) - u(x_0)| &= \left| \int_{\partial B_r(0)} K(x,y) g(y) dS(y) - \int_{\partial B_r(0)} K(x,y)(x_0)| dS(y) \right| \\
&\leq \int_{\partial B_r(0)} K(x,y) |g(y) - g(x_0)| dS(y) \\
&= \int_{\partial B_r(0) \cap B_\delta(x_0)} K(x,y) |g(y) - g(x_0)| dS(y) \\
&\qquad\qquad + \int_{\partial B_r(0) \setminus B_\delta(x_0)} K(x,y) |g(y) - g(x_0)| dS(y) \\
&=: I + J
\end{aligned}$$

Estimating each integral, we have

$$I < \frac{\epsilon}{2} \int_{\partial B_r(0) \cap B_\delta(x_0)} K(x,y)| dS(y) \leq \frac{\epsilon}{2}$$

and for J, we first see that if $|x - x_0| < \frac{\delta}{2}$, then since $y \in \partial B_r(0) \setminus B_\delta(x_0)$, we know that $|y - x_0| \geq \delta$. Thus,

$$|y - x_0| \leq |y - x| + |x - x_0| < |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x_0|$$

Hence, $\frac{1}{|y-x|} \leq \frac{2}{|y-x_0|} \leq \frac{2}{\delta}$, so

$$\begin{aligned}
J &\leq 2\|g\|_{L^\infty(\partial B_r(0))} \int_{\partial B_r(0) \setminus B_\delta(x_0)} K(x,y) dS(y) \\
&= 2\|g\|_{L^\infty(\partial B_r(0))} \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B_r(0) \setminus B_\delta(x_0)} \frac{1}{|y - x|^n} dS(y) \\
&= 2\|g\|_{L^\infty(\partial B_r(0))} \frac{|x_0|^2 - |x|^2}{n\alpha(n) |x_0|} \int_{\partial B_r(0) \setminus B_\delta(x_0)} \frac{1}{|y - x|^n} dS(y) && (|x_0| = r) \\
&\leq 2\|g\|_{L^\infty(\partial B_r(0))} \frac{(|x_0| - |x|) 2 \|x_0|}{n\alpha(n) |x_0|} \int_{\partial B_r(0) \setminus B_\delta(x_0)} \frac{1}{|y - x|^n} dS(y) \\
&\leq 2^2 \|g\|_{L^\infty(\partial B_r(0))} \frac{(|x_0| - |x|)}{n\alpha(n)} \int_{\partial B_r(0) \setminus B_\delta(x_0)} \frac{1}{|y - x|^n} dS(y) \\
&\leq 2^2 \|g\|_{L^\infty(\partial B_r(0))} \frac{(|x_0| - |x|)}{n\alpha(n)} \int_{\partial B_r(0)} \frac{2^n}{\delta^n} dS(y) && \text{(by above)} \\
&= 2^{n+2} \|g\|_{L^\infty(\partial B_r(0))} \frac{(|x_0| - |x|)}{n\alpha(n) \delta^n} n\alpha(n) r^{n-1} \\
&= \frac{2^{n+2} \|g\|_{L^\infty(\partial B_r(0))} r^{n-1}}{\delta^n} (|x_0| - |x|)
\end{aligned}$$

so further assuming that $|x_0 - x| < \frac{\epsilon\delta^n}{2^{n+3}\|g\|_{L^\infty(\partial B_r(0))} r^{n-1}}$, we have

$$J < \frac{\epsilon}{2}$$

Thus,

$$|u(x) - u(x_0)| < I + J < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Evans 2.5.9

Let u be a solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show Du is not bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n) - u(0)}{\lambda}$).

Proof. Using Poisson's formula for the half-space, we have

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dS(y)$$

Let $M > 0$ be a bound on g. By the hint above and noting that $u(0) = 0$,

$$\begin{aligned} \frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{2\lambda}{\lambda n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dS(y) \\ &= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap \{|y| \leq 1\}} \frac{|y|}{|\lambda e_n - y|^n} dS(y) + \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \setminus \{|y| \leq 1\}} \frac{g(y)}{|x-y|^n} dS(y) \\ &\geq \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap \{|y| \leq 1\}} \frac{|y|}{|\lambda e_n - y|^n} dS(y) - \frac{2M}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \setminus \{|y| \leq 1\}} \frac{1}{|x-y|^n} dS(y) \end{aligned}$$

We see that the second integral above is bounded since $n \geq 2$. (The $n = 1$ case is trivial since we integrate over a single point.) Now note that for $y \in \partial\mathbb{R}^n_+$, we must have $y_n = 0$ and for $y \in \{|y| \leq 1\}$, we must have $y_i \leq 1$ for $1 \leq i \leq n$. Thus,

$$\begin{aligned} \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap \{|y| \leq 1\}} \frac{|y|}{|\lambda e_n - y|^n} dS(y) &\geq \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap \{|y| \leq 1\}} \frac{|y|}{(n + \lambda^2)^{n/2}} dS(y) \\ &= \frac{2}{n\alpha(n)(n+\lambda^2)^{n/2}} \int_{\partial\mathbb{R}^n_+ \cap \{|y| \leq 1\}} |y| dS(y) \end{aligned}$$

which goes to $+\infty$ as $\lambda \to 0$. Thus,

$$\lim_{\lambda \to 0} \frac{u(\lambda e_n) - u(0)}{\lambda} = +\infty$$

so $\frac{\partial u}{\partial x_n}$ diverges near 0. Thus, Du cannot be bounded near 0. □

Evans 2.5.10

(Reflection Principle)

(a) Let Ω^+ denote the open half-ball,

$$\Omega^+ = \{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$$

Assume $u \in C^2(\overline{\Omega^+})$ is harmonic in Ω^+, with $u = 0$ on $\partial\Omega^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in \Omega = B_1(0)$. Prove $v \in C^2(\Omega)$ and thus, v is harmonic within Ω.

(b) Now assume only that $u \in C^2(\Omega^+) \cap C(\overline{\Omega^+})$ is harmonic. Show that v is harmonic only in Ω. (Hint: Poisson's formula for the ball.)

Proof.

(a) We see that $v \in C^2(\overline{\Omega^+})$ and $v \in C^2(\Omega \backslash \overline{\Omega^+})$ by definition since $u \in C^2(\overline{\Omega^+})$. Thus, we see that

$$\begin{aligned} \lim_{x_n \to 0^+} \partial_{x_i x_j} v(x_1, \ldots, x_n) &= \partial_{x_i x_j} v(x_1, \ldots, x_{n-1}, 0) \qquad (v \in C^2) \\ &= \partial_{x_i x_j} u(x_1, \ldots, x_{n-1}, 0) \\ &= \lim_{x_n \to 0^-} \partial_{x_i x_j}[u(x_1, \ldots, x_{n-1}, -x_n)] \end{aligned}$$

In the last equality above, we see that

$$\partial_{x_i x_j}[u(x_1, \ldots, -x_n)] = -\lim_{x_n \to 0^-} \partial_{x_i x_j} u(x_1, \ldots, -x_n) = \lim_{x_n \to 0^-} \partial_{x_i x_j} v(x_1, \ldots, x_n)$$

for the case where either i or j equals n. If $i, j < n$, then we know that $u(x) = 0$ for

$$x \in \partial\Omega^+ \cap \{x_n = 0\} = \{x \in \mathbb{R}^n : |x| \leq 1, x_n = 0\}$$

Thus, $\partial_{x_i} u(x) = 0$ for $1 \leq i < n$, and hence $\partial_{x_i x_j} u(x) = 0$ for $1 \leq j < n$. Thus, in this case,

$$\lim_{x_n \to 0^+} \partial_{x_i x_j} v(x_1, \ldots, x_n) = 0 = \lim_{x_n \to 0^-} \partial_{x_i x_j} v(x_1, \ldots, x_n)$$

Finally, for the case where $i = j = n$, we know that $\Delta u = 0$ since u is harmonic and since $\partial_{x_i x_i} u(x) = 0$ for $1 \leq i < n$, then we must have that $\partial_{x_n x_n} u(x) = 0$ as well. Thus, $v \in C^2(\Omega)$ and v is harmonic.

(b) Using Poisson's formula for the ball, we'll define the function

$$w(x) := \begin{cases} \frac{1-|x|^2}{n\alpha(n)r} \int_{\partial\Omega} \frac{v(y)}{|x-y|^n} dS(y) & x \in \Omega \\ v(x) & x \in \partial\Omega \end{cases}$$

Then we first make the observation that for $x \in \Omega \cap \{x_n = 0\}$,

$$\begin{aligned}
w(x) &= \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega}\frac{v(y)}{|x-y|^n}dS(y) \\
&= \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega\cap\{y_n=0\}}\frac{v(y)}{|x-y|^n}dS(y) \\
&\qquad + \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega\cap\{y_n>0\}}\frac{v(y)}{|x-y|^n}dS(y) \\
&\qquad + \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega\cap\{y_n<0\}}\frac{v(y)}{|x-y|^n}dS(y) \\
&= 0 + \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega\cap\{y_n>0\}}\frac{u(y_1,\dots,y_n)}{|x-y|^n}dS(y) \\
&\qquad + \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega\cap\{y_n<0\}}\frac{-u(y_1,\dots,y_{n-1},-y_n)}{|x-y|^n}dS(y)
\end{aligned}$$

Now, we note that $(x_n - y_n)^2 = (x_n + y_n)^2$ iff $x_n = 0$, so using the reflection $y \mapsto \tilde{y}$ where $\tilde{y} = (y_1, \dots, y_{n-1}, -y_n)$, then

$$\begin{aligned}
w(x) &= \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega\cap\{y_n>0\}}\frac{u(y_1,\dots,y_n)}{|x-y|^n}dS(y) \\
&\qquad + \frac{1-|x|^2}{n\alpha(n)r}\int_{\partial\Omega\cap\{y_n>0\}}\frac{-u(y_1,\dots,y_n)}{|x-y|^n}dS(y) \\
&= 0
\end{aligned}$$

Thus, we have that $w = v$ on $\Omega \cap \{x_n = 0\}$, and $w = v$ on $\partial\Omega$. Moreover, since $v \in C^2(\Omega^+) \cap C(\overline{\Omega^+})$ is harmonic, then we may apply the maximum principle on $w - v$ on Ω^+, to get that

$$\max_{\Omega^+} w - v = \max_{\partial\Omega^+} w - v = 0 \qquad \text{and} \qquad \min_{\Omega^+} w - v = \min_{\partial\Omega^+} w - v = 0$$

which, when combined, gives

$$\max_{\Omega^+} |w - v| = 0 \qquad \Longrightarrow \qquad w = v \text{ in } \overline{\Omega^+}$$

Similarly, we can show that $w = v$ in $\overline{\Omega}\backslash\overline{\Omega^+}$. Therefore, v is harmonic on all of Ω.

□

Evans 2.5.12

Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show $v(x,t) = x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

Proof.

(a) This is almost trivial by direct computation,

$$[u_\lambda(x,t)]_t = \lambda^2 u_t(x,t) \qquad \Delta[u_\lambda(x,t)] = \lambda^2 \Delta u(x,t)$$

(b) We notice that

$$\partial_\lambda[u_\lambda(x,t)] = x \cdot Du(\lambda x, \lambda^2 t) + 2\lambda t u_t(\lambda x, \lambda^2 t)$$

and so

$$v(x,t) = [u_\lambda(x,t)]_\lambda \qquad \text{for } \lambda = 1$$

and since u is smooth, we can commute differential operators to get

$$\begin{aligned} v_t - \Delta v = (\partial_t - \Delta)[v] &= (\partial_t - \Delta)(\partial_\lambda)[u_\lambda] \\ &= \partial_\lambda(\partial_t - \Delta)[u_\lambda] \\ &= \partial_\lambda[0] = 0 \end{aligned}$$

□

Evans 2.5.13

Assume $n = 1$ and $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$.

(a) Show

$$u_t = u_{xx} \qquad \text{iff} \qquad v'' + \frac{z}{2}v' = 0$$

and show that the general solution of the ODE above is

$$v(z) = c_1 \int_0^z e^{-\frac{s^2}{4}} ds + c_2$$

(b) Differentiate $u(x,t) = v\left(\frac{x}{\sqrt{t}}\right)$ w.r.t. x and select the constant c properly to obtain the fundametal solution Φ for $n = 1$. Explain why this procedure produces the fundamental solution. (Hint: What is the initial condition for u?)

Proof.

(a) By direct computation,

$$u_t = v'\left(\frac{x}{\sqrt{t}}\right)\left(-\frac{x}{2t^{3/2}}\right) \qquad u_{xx} = v''\left(\frac{x}{\sqrt{t}}\right)\frac{1}{t}$$

Equating the two and letting $z = \frac{x}{\sqrt{t}}$, we have

$$\begin{aligned} v'(z)\left(-\frac{z}{2t}\right) &= v''(z)\frac{1}{t} \\ v'' + \frac{z}{2}v' &= 0 \end{aligned}$$

and solving the above ODE, we have

$$\begin{aligned} \frac{v''}{v'} &= -\frac{z}{2} \\ \ln|v'| &= -\frac{z^2}{4} + c_1 \\ v' &= c_1 e^{-\frac{z^2}{4}} \\ v(z) &= c_1 \int_0^z e^{-\frac{s^2}{4}} ds + c_2 \end{aligned}$$

(b) Differentiating w.r.t. x, we have

$$u_x(x,t) = \frac{c_1}{\sqrt{t}} e^{-\frac{x^2}{4t}}$$

and we notice that $c_1 = \frac{1}{\sqrt{4\pi}}$ gives the fundamental solution for $n = 1$.

□

Evans 2.5.14

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $c \in \mathbb{R}$.

Proof. Define $v(x,t) := u(x,t)e^{ct}$, then we see that

$$\begin{aligned} v_t &= u_t e^{ct} + cue^{ct} \\ \Delta v &= \Delta u e^{ct} \end{aligned}$$

so

$$v_t - \Delta v = (u_t - \Delta u + cu)e^{ct} = fe^{ct}$$

and

$$v(x,0) = u(x,0) = g$$

Thus, v solves the heat equation so we may use the formula for the inhomogeneous initial value solution:

$$v(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)dy + \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)dyds$$

Thus, multiplying by e^{-ct} above gives the solution $u(x,t)$ to the original equation. □

Evans 2.5.15

Given $g : [0,\infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s)ds$$

for a solution of the initial/boundary-value problem,

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0,\infty) \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.)

Proof. Defining $v(x,t) := u(x,t) - g(t)$ for $x \geq 0$ and extending to $x < 0$ by odd reflection, we have

$$v(x,t) = \begin{cases} u(x,t) - g(t) & x \geq 0 \\ -u(-x,t) + g(t) & x < 0 \end{cases}$$

$$v_t(x,t) = \begin{cases} u_t(x,t) - g'(t) & x \geq 0 \\ -u_t(-x,t) + g'(t) & x < 0 \end{cases}$$

$$v_{xx}(x,t) = \begin{cases} u_{xx}(x,t) & x \geq 0 \\ -u_{xx}(-x,t) & x < 0 \end{cases}$$

Thus, we form the following initial/boundary-value problem

$$\begin{cases} v_t - v_{xx} = \begin{cases} -g'(t) & x \geq 0 \\ g'(t) & x < 0 \end{cases} \\ v(x,0) = 0 & x \neq 0 \\ v(0,t) = 0 & t \in (0,\infty) \end{cases}$$

which takes the form of the heat equation. Thus using the formula for its solution, we have

$$\begin{aligned}
v(x,t) &= \int_0^t \left(\int_{\mathbb{R}_-} \Phi(x-y,t-s)g'(s)dy - \int_{\mathbb{R}_+} \Phi(x-y,t-s)g'(s)dy \right) ds \\
&= \int_0^t \left(2\int_{\mathbb{R}_-} \Phi(x-y,t-s)g'(s)dy - g'(s)\int_{\mathbb{R}} \Phi(x-y,t-s)dy \right) ds \\
&= \int_0^t \left(2g'(s)\int_{\mathbb{R}_-} \Phi(x-y,t-s)dy - g'(s) \right) ds \qquad \left(\textstyle\int_{\mathbb{R}} \Phi(y,t)dy = 1 \text{ for any } t\right) \\
&= \int_0^t 2g'(s)\int_{\mathbb{R}_-} \Phi(x-y,t-s)dyds - g(t) - g(0) \\
&= -g(t) + \int_0^t \frac{g'(s)}{\sqrt{\pi}\sqrt{t-s}} \int_{\mathbb{R}_-} e^{-\frac{|x-y|^2}{4(t-s)}} dyds
\end{aligned}$$

Since $v(x,t) = u(x,t) - g(t)$, then

$$\begin{aligned}
u(x,t) &= \int_0^t \frac{g'(s)}{\sqrt{\pi}\sqrt{t-s}} \int_{-\infty}^0 e^{-\frac{|x-y|^2}{4(t-s)}} dyds \\
&= \int_0^t \frac{g'(s)}{\sqrt{\pi}} \left(\int_x^\infty \frac{1}{\sqrt{t-s}} e^{-\frac{z^2}{4(t-s)}} dz \right) ds \qquad (z = x-y)
\end{aligned}$$

Integrating by parts in s, we have

$$\begin{aligned}
u(x,t) &= \left[\frac{g(s)}{\sqrt{\pi}} \int_x^\infty \frac{1}{\sqrt{t-s}} e^{-\frac{z^2}{4(t-s)}} dz \right]_{s=0}^{s=t} \\
&\qquad - \int_0^t \frac{g(s)}{\sqrt{\pi}} \left(\int_x^\infty \frac{1}{2}(t-s)^{-3/2} e^{-\frac{z^2}{4(t-s)}} - \frac{z^2}{4(t-s)^{5/2}} e^{-\frac{z^2}{4(t-s)}} dz \right) ds \\
&= -\int_0^t \frac{g(s)}{\sqrt{\pi}} \int_x^\infty \frac{1}{2(t-s)^{3/2}} e^{-\frac{z^2}{4(t-s)}} dzds \\
&\qquad + \int_0^t \frac{g(s)}{\sqrt{\pi}} \int_x^\infty \frac{z}{2(t-s)^{3/2}} \frac{d}{dz}\left[e^{-\frac{z^2}{4(t-s)}} \right] dzds \\
&=: I + J
\end{aligned}$$

Integrating J by parts in z, we have

$$\begin{aligned}
J &= \int_0^t \frac{g(s)}{\sqrt{\pi}} \left(\left[\frac{z}{2(t-s)^{3/2}} e^{-\frac{z^2}{4(t-s)}} \right]_x^\infty - \int_x^\infty \frac{1}{2(t-s)^{3/2}} e^{-\frac{z^2}{4(t-s)}} dz \right) ds \\
&= \int_0^t \frac{g(s)}{\sqrt{\pi}} \left(\frac{-x}{2(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} - \int_x^\infty \frac{1}{2(t-s)^{3/2}} e^{-\frac{z^2}{4(t-s)}} dz \right) ds \\
&= -\frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(s)}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} ds - I
\end{aligned}$$

Thus,

$$u(x,t) = -\frac{x}{\sqrt{4\pi}} \int_0^t \frac{g(s)}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} ds$$

□

Evans 2.5.16

Give a direct proof that if Ω is bounded and $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ solves the heat equation $u_t - \Delta u = 0$, then

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u$$

(Hint: Define $u_\epsilon := u - \epsilon t$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\overline{\Omega}_T$ at a point in Ω_T)

Proof. Let $u_\epsilon := u - \epsilon t$, $\epsilon > 0$. We first note that if u attains its maximum at a point $(x^0, t_0) \in \Omega_T$, then

$$u_\epsilon(x^0, t_0) = u(x^0, t_0) - \epsilon t_0 \geq u(x,t) - \epsilon t_0 \qquad \text{for all } (x,t) \in \overline{\Omega}_T$$

Taking $\epsilon \to 0$, we have

$$u_\epsilon(x^0, t_0) \geq u(x,t) \geq u(x,t) - \epsilon t = u_\epsilon(x,t) \qquad \text{for all } (x,t) \in \overline{\Omega}_T$$

Thus showing u_ϵ attains its max in Ω_T. Thus, by contrapositive, it suffices to show that u_ϵ cannot attain its max in Ω_T.

Indeed if u_ϵ attains its max at $(x^0, t_0) = (x_1^0, \ldots, x_n^0, t_0) \in \Omega_T$, then we first observe that

$$[u_\epsilon]_t - \Delta u_\epsilon = u_t - \epsilon - \Delta u = -\epsilon < 0$$

Now define $\pi_j : \mathbb{R}^{n+1} \to \mathbb{R}$ as the j-th coordinate map, i.e.

$$\pi_j(x_1, \ldots, x_j, \ldots, x_{n+1}) = x_j$$

Then for each $1 \leq j \leq n+1$, define the map $f_j : \pi_j(\Omega_T) \to \mathbb{R}$ by

$$f_j(z) = \begin{cases} u_\epsilon(x_1^0, \ldots, x_{j-1}^0, z, x_{j+1}^0, \ldots, x_n^0, t_0) & 1 \leq j \leq n \\ u_\epsilon(x_1^0, \ldots, x_n^0, z) & j = n+1 \end{cases}$$

By definition, we have that $f_j(z)$ attains its max at x_j^0 for $1 \leq j \leq n$ and at t_0 for $j = n+1$, hence $f_j''(z) < 0$ and $f_j'(z) = 0$ at such points. Next, we observe that

$$0 = f_{n+1}'(t_0) = \frac{d}{dz} u_\epsilon(x_1^0, \ldots, x_n^0, z)\bigg|_{z=t_0} = [u_\epsilon(x,t)]_t\bigg|_{(x,t)=(x^0,t_0)}$$

$$0 > f_j''(x_j^0) = \frac{d^2}{dz^2} u_\epsilon(x_1^0, \ldots, x_{j-1}^0, z, x_{j+1}^0, \ldots, x_n^0, t_0) = [u_\epsilon(x,t)]_{x_j x_j}\bigg|_{(x,t)=(x^0,t_0)} \qquad (1 \leq j \leq n)$$

Thus,

$$0 < f'_{n=1}(t_0) - \sum_{j=1}^{n} f''_j(x_j^0) = [u_\epsilon]_t - \sum_{j=1}^{n} [u_\epsilon]_{x_j x_j} = [u_\epsilon]_t - \Delta u_\epsilon < 0$$

a contradiction. Thus, u_ϵ does not attain its maximum in Ω_T. □

Evans 2.5.24

(Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, \;\; u_t = h & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

Suppose g, h have compact support. The kinetic energy

$$k(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x,t) dx$$

and the potential energy is

$$p(t) := \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x,t) dx$$

Prove

(a) $k(t) + p(t)$ is constant in time t.

(b) $k(t) = p(t)$ for all large times t.

Proof.

(a) Observe that

$$\begin{aligned} k(t) + p(t) &= \frac{1}{2} \int_{-\infty}^{\infty} u_t^2 + u_x^2 dx \\ \frac{d}{dt}[k(t) + p(t)] &= \frac{1}{2} \int_{-\infty}^{\infty} 2u_t u_{tt} + 2u_x u_{xt} dx \\ &= \int_{-\infty}^{\infty} u_t u_{tt} - u_{xx} u_t dx && \text{(int. by parts)} \\ &= 0 && \text{(by the PDE)} \end{aligned}$$

(b) Next, we first recall d'Alembert's formula,

$$u(x,t) = \frac{1}{2}\left(g(x+t) - g(x-t)\right) + \frac{1}{2} \int_{x-t}^{x+t} h(y) dy$$

$$u_x(x,t) = \frac{1}{2}(g'(x+t) - g'(x-t)) + \frac{1}{2}(h(x+t) - h(x-t))$$
$$u_t(x,t) = \frac{1}{2}(g'(x+t) + g'(x-t)) + \frac{1}{2}(h(x+t) + h(x-t))$$

Thus,

$$\begin{aligned} k(t) - p(t) &= \frac{1}{2}\int_{-\infty}^{\infty} u_t^2 - u_x^2 dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (u_t - u_x)(u_t + u_x) dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} (g'(x-t) + h(x-t))(g'(x+t) + h(x+t)) dx \end{aligned}$$

Since g, h are compactly supported, then we also have that g' is compactly supported, so choose $M > 0$ such that

$$\text{supp}(g'), \text{supp}(h) \subseteq [-M, M]$$

Then for $t > M$, we'll consider the following cases:

- If $x \geq 0$, then
$$g'(x+t) = h(x+t) = 0 \qquad \text{(since } x+t > M\text{)}$$
so that $k(t) - p(t) = 0$
- If $x < 0$, then
$$x - t < x - M < -M$$
so $g'(x-t) = h(x-t) = 0$, so that $k(t) - p(t) = 0$.

Thus, for every $x \in \mathbb{R}$, $k(t) - p(t) = 0$.

□

4 Part C

Evans 5.10.1

Prove that the Holder space $C^{k,\gamma}(\overline{\Omega})$ is a Banach space for any nonnegative integer k and $0 < \gamma \leq 1$.

Proof. Let α be a multi-index with $|\alpha| = k$. We'll first show that $[\cdot]_{C^{k,\gamma}(\overline{\Omega})}$ is a seminorm.

1. Let $\lambda \in \mathbb{R}$ and $u \in C^{k,\gamma}(\overline{\Omega})$. Then

$$\begin{aligned}
[\lambda u]_{C^{k,\gamma}(\overline{\Omega})} &= \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|D^\alpha[\lambda u](x) - D^\alpha[\lambda u](y)|}{|x-y|} \right\} \\
&= \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ |\lambda| \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|} \right\} \\
&= |\lambda| [u]_{C^{k,\gamma}(\overline{\Omega})}
\end{aligned}$$

2. Let $u, v \in C^{k,\gamma}(\overline{\Omega})$.

$$\begin{aligned}
[u+v]_{C^{k,\gamma}(\overline{\Omega})} &= \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|(u+v)(x) - (u+v)(y)|}{|x+y|} \right\} \\
&\leq \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|u(x) - u(y)| + |v(x) - v(y)|}{|x-y|} \right\} \\
&\leq \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|} \right\} + \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|v(x) - v(y)|}{|x-y|} \right\} \\
&= [u]_{C^{k,\gamma}(\overline{\Omega})} + [v]_{C^{k,\gamma}(\overline{\Omega})}
\end{aligned}$$

Next, defining

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} = \|u\|_{C^k(\overline{\Omega})} + [u]_{C^{k,\gamma}(\overline{\Omega})}$$

we will show that $\|u\|_{C^{k,\gamma}(\overline{\Omega})}$ is a norm.

1. Since $\|\cdot\|_{C^k(\overline{\Omega})}$ is a norm and $[\cdot]_{C^{k,\gamma}(\overline{\Omega})}$ is a seminorm, then we know $\|\lambda u\|_{C^{k,\gamma}(\overline{\Omega})} = |\lambda| \cdot \|u\|_{C^{k,\gamma}(\overline{\Omega})}$ and $\|u+v\|_{C^{k,\gamma}(\overline{\Omega})} \leq \|u\|_{C^{k,\gamma}(\overline{\Omega})} + \|v\|_{C^{k,\gamma}(\overline{\Omega})}$.
2. It is clear that $\|0\|_{C^{k,\gamma}(\overline{\Omega})} = 0$, so suppose now that $\|u\|_{C^{k,\gamma}(\overline{\Omega})} = 0$. Thus,

$$\sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{\Omega})} = 0$$

Particularly, $\|u\|_{C(\overline{\Omega})} = 0$ implies that $u = 0$.

Hence, $\|\cdot\|_{C^{k,\gamma}(\overline{\Omega})}$ is a norm. Now let $\epsilon > 0$ and $(u_n)_{n=1}^{\infty} \subset C^{k,\gamma}(\overline{\Omega})$ be a Cauchy sequence. Then there exists $N \in \mathbb{N}$ such that if $n, m \geq N$ then $\|u_n - u_m\|_{C^{k,\gamma}(\overline{\Omega})} < \epsilon$. Thus, we see that

$$\|u_n\|_{C^{k,\gamma}(\overline{\Omega})} \leq \|u_n - u_N\|_{C^{k,\gamma}(\overline{\Omega})} + \|u_N\|_{C^{k,\gamma}(\overline{\Omega})} < \epsilon + \|u_N\|_{C^{k,\gamma}(\overline{\Omega})} < \infty$$

since $\overline{\Omega}$ is compact. Hence, u_n is bounded, i.e.

$$\|u_n\|_{C^{k,\gamma}(\overline{\Omega})} \leq \max\{\|u_1\|_{C^{k,\gamma}(\overline{\Omega})}, \cdots \|u_N\|_{C^{k,\gamma}(\overline{\Omega})}\}$$

Thus, there exists a convergent subsequence $(u_{n_k})_{k=1}^{\infty}$. Let $\lim_{k\to\infty} u_{n_k} = u$. Next, there exists $N_1, N_2 \in \mathbb{N}$ such that $\|u_n - u_{n_k}\|_{C^{k,\gamma}(\overline{\Omega})} < \epsilon/2$ for $n, n_k \geq N_1$ and $\|u_{n_k} - u\|_{C^{k,\gamma}(\overline{\Omega})} < \epsilon/2$ if $n_k \geq N_2$. Choosing the larger of the two, we have

$$\|u_n - u\|_{C^{k,\gamma}(\overline{\Omega})} \leq \|u_n - u_{n_k}\|_{C^{k,\gamma}(\overline{\Omega})} + \|u_{n_k} - u\|_{C^{k,\gamma}(\overline{\Omega})} < \epsilon.$$

for all $n, n_k \geq \max\{N_1, N_2\}$. Thus, $u_n \to u$. To show that $u \in C^{k,\gamma}(\overline{\Omega})$, we recall that $u_n \in C^{k,\gamma}(\overline{\Omega})$, so there exists $C > 0$ such that

$$|D^\alpha u_n(x) - D^\alpha u_n(y)| < C|x-y|^\gamma$$

Thus, if we choose n sufficiently large so that $\|u - u_n\|_{C^{k,\gamma}(\overline{\Omega})} < \epsilon/2$, we have

$$\begin{aligned}|D^\alpha u(x) - D^\alpha u(y)| &\leq |D^\alpha u(x) - D^\alpha u_n(x)| + |D^\alpha u_n(x) - D^\alpha u_n(y)| + |D^\alpha u_n(y) - D^\alpha u(y)| \\ &\leq 2\|u - u_n\|_{C^{k,\gamma}(\overline{\Omega})} + |D^\alpha u_n(x) - D^\alpha u_n(y)| \\ &< \epsilon + C|x-y|^\gamma\end{aligned}$$

so we have that

$$|D^\alpha u(x) - D^\alpha u(y)| \leq C|x-y|^\gamma < (C+1)|x-y|^\gamma$$

Hence, $u \in C^{k,\gamma}(\overline{\Omega})$. □

Evans 5.2 Example 2

Consider the function

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 \leq x < 2 \end{cases}$$

Show that $f(x)$ does not have a weak derivative.

Solution: Suppose by contradiction that f has a weak derivative g, i.e. $f' = g$ in the weak sense. Then for all test functions, $h \in C_c^\infty([0,2])$, we have that

$$\begin{aligned} \int_0^2 fh' &= -\int_0^2 gh && (g = f') \\ \int_0^1 h' &= -\int_0^2 gh && (\text{Definition of } f) \\ h(1) - h(0) &= -\int_0^2 gh && (\text{FTC}) \\ h(1) &= -\int_0^2 gh && (h \in C_c([0,2])) \end{aligned}$$

Now, consider the sequence $(h_m)_{m=1}^\infty \subset C_c^\infty([0,2])$ where

$$h_m(x) = (2x - x^2)^m$$

Then, we know that $h_m(1) = 1$ for all m and for $x \in [0,2]\backslash\{1\}$, we see that $2x - x^2 \in (0,1)$, so $h_m(x) \to 0$ as $m \to \infty$. Thus,

$$h_m(1) = 1 = -\int_0^2 g(x)(2x - x^2)^m dx$$

Hence, taking $m \to \infty$, we see that

$$1 = \lim_{m\to\infty} -\int_0^2 g(x)(2x - x^2)^m dx = 0$$

a contradiction. Thus, f does not have a weak derivative.

Product Rule for Weak Derivatives

If $f \in L^1_{\text{loc}}(\Omega)$ has a weak partial derivative $f_{x_i} \in L^1_{\text{loc}}(\Omega)$ and $\psi \in C^\infty(\Omega)$, then ψf is weakly differentiable with respect to x_i and

$$(\psi f)_{x_i} = \psi_{x_i} f + \psi(f_{x_i})$$

Proof. Let $\phi \in C_c^\infty(\Omega)$. Then, we know that $(\psi\phi) \in C_c^\infty(\Omega)$, so we may use $\psi\phi$ as the test function for the weak differentiability of f.

$$\begin{aligned} -\int_\Omega f_{x_i}(\psi\phi)dx &= \int_\Omega f(\psi\phi)_{x_i} dx \\ &= \int_\Omega f(\psi_{x_i}\phi + \psi\phi_{x_i})dx && (\text{classical product rule}) \\ &= \int_\Omega (f\psi_{x_i})\phi dx + \int_\Omega (f\psi)\phi_{x_i} dx \\ \int_\Omega (f\psi)\phi_{x_i} dx &= -\int_\Omega (f\psi_{x_i} + f_{x_i}\psi)\phi dx \end{aligned}$$

□

Evans 5.10.2

Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(\Omega)} \leq \|u\|_{C^{0,\beta}(\Omega)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(\Omega)}^{\frac{\gamma-\beta}{1-\beta}}$$

Proof. We first recall that

$$\|u\|_{C^{0,\gamma}(\Omega)} = \|u\|_{C(\Omega)} + [u]_{C^{0,\gamma}(\Omega)}$$

and we'll let $p := \frac{1-\gamma}{1-\beta}$ and $q := \frac{\gamma-\beta}{1-\beta}$ and we see that $p + q = 1$. Now, we see that

$$\begin{aligned}
\|u\|_{C^{0,\gamma}(\Omega)} &= \|u\|_{C(\Omega)}^{p+q} + [u]_{C^{0,\gamma}(\Omega)} \\
&= \|u\|_{C(\Omega)}^{p}\|u\|_{C(\Omega)}^{q} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left(\frac{|u(x)-u(y)|^{p+q}}{|x-y|^{\gamma}} \right) \\
&= \|u\|_{C(\Omega)}^{p}\|u\|_{C(\Omega)}^{q} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left(\frac{|u(x)-u(y)|^{p}\ |u(x)-u(y)|^{q}}{|x-y|^{q}\left(|x-y|^{\beta}\right)^{p}} \right) && (q + p\beta = \gamma) \\
&\leq \|u\|_{C(\Omega)}^{p}\|u\|_{C(\Omega)}^{q} + [u]_{C^{0,\beta}(\Omega)}^{p}[u]_{C^{0,1}(\Omega)}^{q}
\end{aligned}$$

Now let $a := \|u\|_{C(\Omega)}$, $b := [u]_{C^{0,\beta}(\Omega)}$, and $c := [u]_{C^{0,1}(\Omega)}$. Then

$$\begin{aligned}
\|u\|_{C^{0,\gamma}(\Omega)} &\leq a^p a^q + b^p c^q \\
&= (a+b)^p \left(\frac{a^p a^q}{(a+b)^p} + \frac{b^p c^q}{(a+b)^p} \right) && (\text{force } (a+b)^p) \\
&= (a+b)^p \left(\frac{a^{1-q} a^q}{(a+b)^{1-q}} + \frac{b^{1-q} c^q}{(a+b)^{1-q}} \right) && (\text{convert to } q \text{ exponent}) \\
&= (a+b)^p \left(\frac{a}{a+b} \left(\frac{a(a+b)}{a} \right)^q + \frac{b}{a+b} \left(\frac{c(a+b)}{b} \right)^q \right) && (\text{collect terms with } q) \\
&\leq (a+b)^p (a+c)^q && (\text{concavity of } x^q,\ q \in (0,1)) \\
&= \|u\|_{C^{0,\beta}(\Omega)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(\Omega)}^{\frac{\gamma-\beta}{1-\beta}}
\end{aligned}$$

□

Evans 5.10.4

Assume $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$.

(a) Show that u is equal a.e. to an absolutely continuous function and u' (which exists a.e.) belongs to $L^p(0,1)$.

(b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p dt \right)^{1/p}$$

for a.e. $x, y \in [0,1]$.

Proof.

(a) Since u' exists a.e. and $u' \in L^p(0,1)$, then by Holder's inequality, $u' \in L^1(0,1)$, so let $v(x) := \int_0^x u'(y)dy$ for $x \in (0,1)$. Then by the fundamental theorem of calculus for Lebesgue integrals, we know that v is absolutely continuous on $(0,1)$. Now consider a test function $\phi \in C_c^\infty(0,1)$ and observe that

$$\begin{aligned}
\int_0^1 (v-u)\phi' dy &= \int_0^1 \left(\int_0^y u'(x)dx \right) \phi'(y)dy - \int_0^1 u(y)\phi'(y)dy \\
&= -\int_0^1 u'(y)\phi(y)dy + \int_0^1 u'(y)\phi(y)dy \\
&= 0
\end{aligned}$$

Since this holds for all $\phi \in C_c^\infty(0,1)$, then $v = u$ a.e.

(b) By (a), since u is absolutely continuous a.e., we may apply FTC, to get

$$\begin{aligned}
|u(x) - u(y)| = \left| \int_x^y u'(t)dt \right| &\le \int_x^y |u'(t)|dt \\
&\le \|u\|_{L^1(x,y)} && \text{(assume } x < y\text{)} \\
&\le |x-y|^{1-\frac{1}{p}} \|u'\|_{L^p(x,y)} \\
&\le |x-y|^{1-\frac{1}{p}} \|u'\|_{L^p(0,1)}
\end{aligned}$$

□

Evans 5.10.7

Assume that Ω is bounded open subset of $\mathbb{R}^n$ and there exists a smooth vector field $\alpha : \Omega \to \mathbb{R}^n$ such that $\alpha \cdot \nu \ge 1$ along $\partial\Omega$, where ν denotes the usual outward unit normal. Assume $1 \le p < \infty$.
Apply the Gauss-Green theorem to $\int_{\partial\Omega} |u|^p \alpha \cdot \nu dS$, to derive a new proof of the trace inequality

$$\int_{\partial\Omega} |u|^p dS \le C \int_\Omega |Du|^p + |u|^p dy$$

for all $u \in C(\overline{\Omega})$.

Proof. Since $u \in C(\overline{\Omega})$, applying the Gauss-Green theorem, we have

$$\begin{aligned}
\int_{\partial\Omega} |u|^p dS &\leq \int_{\partial\Omega} |u|^p \alpha \cdot \nu dS && (\alpha \cdot \nu \geq 1) \\
&\leq \int_{\Omega} \nabla \cdot (|u|^p \alpha) dy && \text{(Gauss-Green)} \\
&= \int_{\Omega} |u|^p (\nabla \cdot \alpha) + \nabla(|u|^p) \cdot \alpha dy \\
&= \int_{\Omega} |u|^p (\nabla \cdot \alpha) + p|u|^{p-1} \operatorname{sgn}(u)(Du \cdot \alpha) dy \\
&\leq C \int_{\Omega} |u|^p + p|u|^{p-1}|Du| dy && (\alpha \text{ smooth on } \Omega \text{ bounded}) \\
&\leq C \int_{\Omega} |u|^p + p\left(\frac{(|u|^{p-1})^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{|Du|^p}{p}\right) dy && \text{(Young's inequality)} \\
&= C \int_{\Omega} |u|^p + (p-1)|u|^p + |Du|^p dy \\
&\leq C \int_{\Omega} |u|^p + |Du|^p dy
\end{aligned}$$

□

Evans 5.10.8

Let Ω be bounded, with a C^1 boundary. Show that a typical function $u \in L^p(\Omega)$ $(1 \leq p < \infty)$ does not have a trace on $\partial\Omega$. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(\Omega) \to L^p(\partial\Omega)$$

such that $Tu = u\big|_{\partial\Omega}$ whenever $u \in C(\overline{\Omega}) \cap L^p(\Omega)$

Proof. Suppose there exists such a T. Then consider the sequence

$$u_n(x) = e^{-n\cdot \operatorname{dist}(x, \partial\Omega)}, \qquad x \in \Omega$$

Then it is clear that $u_n(x) \in (0, 1]$ for all $n \in \mathbb{N}$ and $x \in \Omega$. Thus, $u_n \in L^2(\Omega)$. For $x \in \partial\Omega$, $u_n(x) = 1$ for all n, and if $x \in \Omega$, then $u_n(x) \to 0$ pointwise as $n \to \infty$, so by the dominated convergence theorem, we have that

$$\|u_n\|_{L^2(\Omega)}^2 \to 0$$

By definition, since T is bounded, there must exist some $C > 0$ such that

$$\|Tu_n\|_{L^2(\partial\Omega)} \leq C\|u_n\|_{L^2(\Omega)}$$

but since $u_n \equiv 1$ on $\partial\Omega$, then $Tu_n \equiv 1$, so for sufficiently large n we have

$$\|1\|_{L^2(\partial\Omega)} = \|Tu_n\|_{L^2(\partial\Omega)} \leq C\|u_n\|_{L^2(\Omega)} < \|1\|_{L^2(\partial\Omega)}$$

a contradiction, so no such T may exist. □

Evans 5.10.9

Integrate by parts to prove the interpolation inequality:

$$\|Du\|_{L^2} \leq C\|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2}$$

for all $u \in C_c^\infty(\Omega)$. Assume Ω is bounded, $\partial\Omega$ is smooth, and prove the same inequality for $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. For $u \in C_c^\infty(\Omega)$,

$$\begin{aligned}
\|Du\|_{L^2}^2 &= \int_\Omega |Du|^2 dx \\
&= \int_{\partial\Omega} u \cdot Du \cdot \eta dS(x) - \int_\Omega u\Delta u dx && \text{(int. by parts)} \\
&= 0 - \int_\Omega u\Delta u dx \\
&\leq \int_\Omega |u||\Delta u| dx && (u \in C_c(\Omega)) \\
&\leq \int_\Omega |u||D^2u| dx && (\Delta u = \text{tr}(D^2u)) \\
&\leq \|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2} && \text{(Holder's inequality)}
\end{aligned}$$

Now assume u is only in $H^2(\Omega) \cap H_0^1(\Omega)$. Then since $W^{n,p} \subseteq W^{m,p}$ for $n \geq m$, then we know that $u \equiv 0$ on $\partial\Omega$ in the trace sense (Trace-zero theorem). Thus, the same calculation as above holds with the only changes being Du in the weak sense and the integral over the boundary is zero because of trace-zero. □

Evans 5.10.11

Suppose Ω is connected and $u \in W^{1,p}(\Omega)$ satisfies

$$Du = 0 \qquad \text{a.e. in } \Omega$$

Prove u is constant a.e. in Ω.

Proof. Let η_ϵ be the standard mollifier and define

$$u^\epsilon := u * \eta_\epsilon \qquad \text{in } \Omega_\epsilon$$

Then since

$$D[u^\epsilon] = D[u * \eta_\epsilon] = Du * \eta_\epsilon = 0 * \eta_\epsilon = 0 \qquad \text{in } \Omega_\epsilon$$

Since u^ϵ is smooth, then u^ϵ must be constant a.e. in Ω_ϵ. Moreover since $u^\epsilon \to u$ a.e., then u must also be constant a.e. in Ω_ϵ. Thus, taking $\epsilon \to 0$ gives u constant a.e. in Ω. □

Evans 5.10.14

Verify that if $n > 1$, the unbounded function $u = \log\log\left(1 + \frac{1}{|x|}\right)$ belongs to $W^{1,n}(\Omega)$, for $\Omega = B_1(0)$.

Proof. We first calculate

$$\begin{aligned} u_{x_i} &= \frac{1}{\ln(1+1/|x|)} \frac{1}{1+1/|x|} \frac{-1}{|x|^2} \frac{x_i}{|x|} \\ &= \frac{1}{\ln(1+1/|x|)} \frac{-x_i}{|x|+1} \frac{1}{|x|^2} \\ |Du| &= \frac{1}{\ln(1+1/|x|)} \frac{-1}{|x|+1} \frac{1}{|x|} \end{aligned}$$

We'll first show that $Du \in L^n(B_1(0))$. Indeed,

$$\begin{aligned} \|Du\|_{L^n(B(0,1))} &= \int_{B(0,1)} \left[\left(\frac{1}{\ln(1+\frac{1}{|x|})} \right) \left(\frac{1}{|x|+1} \right) \frac{1}{|x|} \right]^n dx \\ &= \int_0^1 \int_{\partial B(0,r)} \frac{1}{\ln^n(1+1/r)} \frac{1}{(r+1)^n} \frac{1}{r^n} dS(x) dr && \text{(polar coordinates)} \\ &= \int_0^1 \frac{1}{\ln^n(1+1/r)} \frac{1}{(r+1)^n} \frac{1}{r^n} (n\alpha(n) r^{n-1}) dr \\ &= n\alpha(n) \int_0^1 \frac{1}{\ln^n(1+1/r)} \frac{1}{(r+1)^n} \frac{1}{r} dr \\ &\leq n\alpha(n) \int_0^1 \frac{1}{\ln^n(1+1/r)} \frac{1}{r} dr && \left(\tfrac{1}{r+1} \leq 1\right) \\ &= n\alpha(n) \int_{\ln(2)}^\infty \frac{1}{\ln^n(1+1/r)} \frac{1}{r} r(1+r) du && \begin{cases} u = \ln(1+1/r) \\ dr = -r(1+r)du \end{cases} \\ &= n\alpha(n) \int_{\ln(2)}^\infty \frac{1}{u^n} \left(1 + \frac{1}{e^u - 1}\right) du \\ &< n\alpha(n) \int_{\ln(2)}^\infty \frac{1}{u^n} du \\ &< \infty && \text{(since } n > 1) \end{aligned}$$

Thus, $Du \in L^n(\Omega)$. Next, we have that

$$\begin{aligned}
\|u\|_{L^n(B(0,1))} &= \int_{B(0,1)} \left|\ln\left(\ln\left(1+\frac{1}{|x|}\right)\right)\right|^n \\
&= n\alpha(n)\int_0^1 r^{n-1}\left|\ln\left(\ln\left(1+\frac{1}{r}\right)\right)\right|^n dr && \text{(polar coordinates)} \\
&= n\alpha(n)\int_{\ln(2)}^{\infty} r^{n-1}\left|\ln\left(1+\frac{1}{r}\right)\right|^n dr \\
&= n\alpha(n)\int_{\ln(2)}^{\infty} r^{n-1}\left|\ln\left(1+\frac{1}{r}\right)\right|^n r(1+r)du && \begin{cases} u = \ln(1+1/r) \\ dr = -r(1+r)du \end{cases} \\
&= n\alpha(n)\int_{\ln(2)}^{\infty} \left(\frac{1}{e^u-1}\right)^n u^n \left(1+\frac{1}{e^u-1}\right) du \\
&\le 2\int_{\ln(2)}^{\infty}\left(\frac{u}{e^u-1}\right)^n du && \left(\tfrac{1}{e^u-1}\le 2\right) \\
&\le 2\int_{\ln(2)}^{\infty}\left(\frac{u}{e^u-\frac{1}{2}e^u}\right)^n du \\
&\le 2^{n+1}\int_{\ln(2)}^{\infty}\frac{u^n}{e^{nu}}du \\
&< \infty && \text{(Integration by parts } n \text{ times)}
\end{aligned}$$

Thus, $u \in L^n(\Omega)$ as well. Finally, we want to confirm that Du is indeed the weak derivative of u, but we know that u is pointwise differentiable in the classical sense away from $x = 0$, so for $\phi \in C_c^\infty(\Omega)$, observe that

$$\begin{aligned}
\int_{\Omega\setminus B_\epsilon(0)} u\phi' dx &= -\int_{\Omega\setminus B_\epsilon(0)} Du\phi dx + \int_{\partial B_\epsilon(0)} u\phi dS(x) + \int_{\partial\Omega} u\phi dS(x) \\
&= -\int_{\Omega\setminus B_\epsilon(0)} Du\phi dx + \int_{\partial B_\epsilon(0)} u\phi dS(x) && \text{(since } \phi \in C_c(\Omega))
\end{aligned}$$

Taking the last integral, we see that

$$\begin{aligned}
\int_{\partial B_\epsilon(0)} u\phi dS(x) &= \int_{\partial B_\epsilon(0)} \ln\left(\ln\left(1+\frac{1}{|x|}\right)\right)\phi(x)dS(x) \\
&\le \|\phi\|_{L^\infty(\partial B_\epsilon(0))}\int_{\partial B_\epsilon(0)} \ln\left(1+\frac{1}{|x|}\right) dS(x) \\
&= \|\phi\|_{L^\infty(\partial B_\epsilon(0))} n\alpha(n)\ln\left(1+\frac{1}{\epsilon}\right)\epsilon^{n-1}
\end{aligned}$$

and since $n > 1$ and we know that

$$\lim_{\epsilon\to 0^+}\epsilon\ln\left(1+\frac{1}{\epsilon}\right)\to 0 \qquad \text{(by L'hopital's)}$$

then we may take $\epsilon \to 0^+$ to find

$$\int_\Omega u\phi' dx = -\int_\Omega Du\phi dx$$

□

Evans 5.10.15

Fix $\alpha > 0$ and let $\Omega = B_1(0)$. Show that there exists a constant C, depending only on n and α, such that

$$\int_\Omega u^2 dx \leq C \int_U |Du|^2 dx$$

provided

$$|\{x \in \Omega : u(x) = 0\}| \geq \alpha \qquad u \in H^1(\Omega)$$

Proof. Using Poincare's inequality, we have

$$\begin{aligned}
C\int_\Omega |Du|^2 dx &\geq \int_\Omega (u - (u)_\Omega)^2 dx \\
&= \int_\Omega u^2 - 2u(u)_\Omega + (u)_\Omega^2 dx \\
&= \int_\Omega u^2 - u(u)_\Omega dx - (u)_\Omega \int_\Omega u dx + (u)_\Omega^2 |\Omega| \\
&= \int_\Omega u^2 - u(u)_\Omega dx - (u)_\Omega \frac{|\Omega|}{|\Omega|} \int_\Omega u dx + (u)_\Omega^2 |\Omega| \\
&= \int_\Omega u^2 - u(u)_\Omega dx - |\Omega|(u)_\Omega^2 + (u)_\Omega^2 |\Omega| \\
&= \int_\Omega u^2 - u(u)_\Omega dx
\end{aligned}$$

Next, we have that

$$\begin{aligned}
\int u(u)_\Omega dx = \frac{1}{|\Omega|}\left(\int_\Omega u dx\right)^2 &\leq \frac{1}{|\Omega|} \|1\|^2_{L^2(\{x\in\Omega: u(x)\neq 0\})} \|u\|^2_{L^2(\{x\in\Omega: u(x)\neq 0\})} && \text{(Holder's ineq.)} \\
&\leq \frac{|\Omega| - \alpha}{|\Omega|} \|u\|^2_{L^2(\{x\in\Omega: u(x)\neq 0\})} && \text{(measure of support of } u\text{)} \\
&= \frac{|\Omega| - \alpha}{|\Omega|} \|u\|^2_{L^2(\Omega)} && \text{(since } u = 0 \text{ outside of its support)}
\end{aligned}$$

Thus, combining both results,

$$\begin{aligned}
C\int_\Omega |Du|^2 dx &\geq \int_\Omega u^2 dx - \frac{|\Omega| - \alpha}{|\Omega|} \|u\|^2_{L^2(\Omega)} \\
&= \left(1 - \frac{|\Omega| - \alpha}{|\Omega|}\right) \|u\|^2_{L^2(\Omega)}
\end{aligned}$$

and since $\alpha \leq |\Omega|$, we may divide it over and we are done. □

Evans 5.10.17

(Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose Ω is bounded and $u \in W^{1,p}(\Omega)$ for some $1 \leq p \leq \infty$. Show

$$v := F(u) \in W^{1,p}(\Omega) \qquad \text{and} \qquad v_{x_i} = F'(u)u_{x_i} \qquad \text{for} \qquad i = 1, \dots, n$$

Proof. We'll first show that $v \in L^p(\Omega)$. Let $(u_m) \subset C^\infty(\Omega)$ be a smooth sequence approximating u. Then

$$\begin{aligned}
\|v\|_{L^p(\Omega)} = \|F(u)\|_{L^p(\Omega)} &\leq \|F(u) - F(u_m)\|_{L^p(\Omega)} + \|F(u_m)\|_{L^p(\Omega)} \\
&= \left(\int_\Omega |F(u) - F(u_m)|^p\right)^{1/p} + \|F(u_m)\|_{L^p(\Omega)} \\
&\leq \left(\int_\Omega C^p|u - u_m|^p\right)^{1/p} + \|F(u_m)\|_{L^p(\Omega)} \qquad (F \text{ Lipschitz}) \\
&= C\|u - u_m\|_{L^p(\Omega)} + \|F(u_m)\|_{L^p(\Omega)} \\
&< \infty
\end{aligned}$$

with the last inequality holding since $u_m \to u$ in L^p and $F \in C^1(\mathbb{R})$, with Ω bounded.

Next, we'll show that $v_{x_i} = F'(u)u_{x_i}$. Using smooth approximation (as shown above in the Lipschitz argument), we know that

$$F(u_m) \to F(u) = v \qquad \text{in } L^p(\Omega)$$

Next, we have that

$$\begin{aligned}
\|F'(u_m)[u_m]_{x_i} - F'(u)u_{x_i}\|_{L^p(\Omega)} &= \|F'(u_m)[u_m]_{x_i} - F'(u_m)u_{x_i} + F'(u_m)u_{x_i} - F'(u)u_{x_i}\|_{L^p(\Omega)} \\
&\leq \|F'(u_m)([u_m]_{x_i} - u_{x_i})\|_{L^p} + \|(F'(u_m) - F'(u))u_{x_i}\|_{L^p} \\
&\leq \|F'\|_{L^\infty(u_m(\Omega))}\|[u_m]_{x_i} - u_{x_i}\|_{L^p} + \|(F'(u_m) - F'(u))u_{x_i}\|_{L^p} \\
&\to 0
\end{aligned}$$

where the first integral goes to 0 by $W^{1,p}$ convergence and the second goes to 0 by the dominated convergence theorem since $F' \in C(\mathbb{R})$. Thus, $F(u_m) \to F(u)$ and $F'(u_m)[u_m]_{x_i} \to F'(u)u_{x_i}$ in $L^p(\Omega)$ so by the uniqueness of the weak derivative, we must have that

$$[F(u)]_{x_i} = F'(u)u_{x_i} \qquad \text{for a.e. } x \in \Omega$$

Last, $Dv = F'(u)Du \in L^p(\Omega)$ since $F' \in C(u(\Omega))$ and $Du \in L^p(\Omega)$. $\square$

Evans 6.6.2

Let

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + cu$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypothesis of the Lax-Milgram theorem, provided $c(x) \geq -\mu$ for all $x \in \Omega$.

Proof. We will first prove that there exists $\alpha > 0$ such that

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

for $u, v \in H_0^1(\Omega)$. Indeed,

$$\begin{aligned}
|B[u,v]| &= \left| \int_\Omega -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} v + cuv \;\, dx \right| \\
&= \left| \int_\Omega \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + cuv \;\, dx \right| && \text{(int. by parts)} \\
&\leq \sup_{1\leq i,j\leq n} \|a^{ij}\|_\infty \int_\Omega |Du||Dv| dx + \|c\|_\infty \int_\Omega |u||v| dx && (a^{ij}, c \text{ bounded}) \\
&\leq \alpha \left(\|DuDv\|_{L^1} + \|uv\|_{L^1}\right) && (\text{take } \alpha \text{ max}) \\
&\leq \alpha \left(\|Du\|_{L^2}\|Dv\|_{L^2} + \|u\|_{L^2}\|v\|_{L^2}\right) && \text{(Holder's ineq.)} \\
&\leq \alpha \|u\|_{H_0^1} \|v\|_{H_0^1} && (\text{since } \|u\|_{L^2}, \|Du\|_{L^2} \leq \|u\|_{H_0^1})
\end{aligned}$$

Next, we'll show that

$$\beta \|u\|^2_{H_0^1(\Omega)} \leq B[u,u]$$

for a certain $\mu > 0$. By uniform ellipticity, there exists $\theta > 0$ such that

$$\begin{aligned}
\theta \int_\Omega |Du|^2 dx &\leq \int_\Omega \sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} \\
&= B[u,u] - \int_\Omega cu^2 dx && (\text{int. by parts on } B[u,u]) \\
&\leq B[u,u] + \mu \int_\Omega u^2 dx && (c(x) \geq -\mu) \\
&\leq B[u,u] + A\mu \int_\Omega |Du|^2 dx && (\text{Estimate on } W_0^k(\Omega)) \\
(\theta - A\mu) \int_\Omega |Du|^2 dx &\leq B[u,u]
\end{aligned}$$

Choosing $0 < \mu < \frac{\theta}{A}$ gives us $\theta - A\mu > 0$ and using the estimate on $W_0^k(\Omega)$ again gives us that

$$\beta\|u\|_{H_0^1}^2 \leq \frac{\theta - A\mu}{2A}\int_\Omega u^2 dx + \frac{\theta - A\mu}{2}\int_\Omega |Du|^2 dx \leq (\theta - A\mu)\int_\Omega |Du|^2 dx \leq B[u,u]$$

where $\beta = \min\left\{\frac{\theta - A\mu}{2A}, \frac{\theta - A\mu}{2}\right\}$. □

Evans 6.6.3

A function $u \in H_0^2(\Omega)$ is a weak solution of this boundary-value problem for the biharmonic equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

provided

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx$$

for all $v \in H_0^2(\Omega)$. Given $f \in L^2(\Omega)$, prove that there exists a unique weak solution for the biharmonic equation.

Proof. In order to invoke Lax-Milgram, we'll prove that the differential operator

$$Lu = -\Delta^2 u$$

satisfies its hypothesis.

1. Observe that

$$\begin{aligned} |B[u,v]| &= \left|\int_\Omega -\Delta^2 u v dx\right| \\ &= \left|\int_\Omega \Delta u \Delta v\right| && \text{(int. by parts and } \tfrac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega) \\ &\leq \int_\Omega |\Delta u \Delta v| dx \\ &\leq \|\Delta u\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)} && \text{(Holder's ineq.)} \\ &\leq \|u\|_{H_0^2(\Omega)}\|v\|_{H_0^2(\Omega)} && \text{(since } \|u\|_{L^2}, \|Du\|_{L^2}, \|\Delta u\|_{L^2} \leq \|u\|_{H_0^2}) \end{aligned}$$

2. Next, we first observe that

$$\begin{aligned} \|u\|_{L^2(\Omega)}^2 &\leq C_1\|Du\|_{L^2(\Omega)} \\ &= C\int_\Omega -u\Delta u dx && \text{(int. by parts)} \\ &\leq C\|u\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} && \text{(Holder's ineq.)} \\ \|u\|_{L^2(\Omega)} &\leq C\|\Delta u\|_{H_0^2(\Omega)} \end{aligned}$$

followed by

$$\begin{aligned} \|Du\|_{L^2(\Omega)}^2 &\leq \|u\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} \\ &\leq C\|Du\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)} && \text{(estimate on } W_0^{1,p}\text{)} \\ \|Du\|_{L^2(\Omega)} &\leq C\|\Delta u\|_{L^2(\Omega)} \end{aligned}$$

Thus, we have that

$$\|\Delta u\|_{L^2(\Omega)}^2 \geq \frac{1}{C}\|Du\|_{L^2(\Omega)}^2 \qquad \text{and} \qquad \|\Delta u\|_{L^2(\Omega)}^2 \geq \frac{1}{C}\|u\|_{L^2(\Omega)}^2$$

Thus, we have

$$B[u,u] = \|\Delta u\|_{L^2}^2 = 3\left(\frac{1}{3}\right)\|\Delta u\|_{L^2}^2 \geq \frac{1}{3}\|\Delta u\|_{L^2}^2 + \frac{1}{3C}\left(\|Du\|_{L^2}^2 + \|u\|_{L^2}^2\right) \geq \beta\|u\|_{H_0^2(\Omega)}$$

by letting $\beta = \min\{1/3, 1/3C\}$.

□

Evans 6.6.4

Assume Ω is connected. A function $u \in H^1(\Omega)$ is a weak solution of Neumann's problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

if

$$\int_\Omega Du \cdot Dv dx = \int_\Omega fv dx$$

for all $v \in H^1(\Omega)$. Suppose $f \in L^2(\Omega)$. Prove that Neumann's problem has a weak solution iff

$$\int_\Omega f dx = 0$$

Proof Outline.

1. Forward direction is trivial, just choose $v \equiv 1$.
2. For the backward direction, we want to invoke Lax-Milgram, but constant functions break the $B[u,u] \geq \beta\|u\|_{H^1}^2$ condition. Other condition is trivial.
3. With the fact that the average of constant functions are themselves, we restrict H^1 to just those that have average equal to zero.
4. Prove this is a closed subset of H^1 under the same norm, thus making it a Hilbert space as well
5. Use Poisson's ineq. to split $\|Du\|_{L^2}^2$ to find $\|u\|_{H^1}^2$

6. Lax-Milgram gives a solution on the restricted Hilbert space. Extend it to Ω by using the hypothesis $\int f dx = 0$.

■

Proof. ($\Rightarrow$) In the forward direction, since we know that

$$\int_\Omega Du \cdot Dv dx = \int_\Omega f v dx \qquad \text{for all } v \in H^1(\Omega)$$

then we simply choose $v \equiv 1 \in H^1(\Omega)$ so that

$$\int_\Omega f dx = \int_\Omega Du \cdot 0 dx = 0$$

($\Leftarrow$) Our goal now is to invoke Lax-Milgram. We first define $Lu = -\Delta u$ and using integration by parts, we see that

$$B[u, v] = \int_\Omega Luv dx = \int_\Omega -\Delta uv dx = \int_\Omega Du \cdot Dv dx \qquad \left(\text{since } \frac{\partial u}{\partial \nu} = 0\right)$$

Thus, for boundedness, we have

$$|B[u, v]| \leq \int_\Omega |Du||Dv| dx \leq \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

Next, for the second condition of Lax-Milgram, we want to show that

$$B[u, u] \geq \beta \|u\|^2_{H^1(\Omega)}$$

for some $\beta > 0$. However, we notice that if u is a constant function $u \equiv \lambda \in \mathbb{R}$, then

$$B[u, u] = \int_\Omega |D\lambda|^2 dx = 0 \qquad \text{but} \qquad \|\lambda\|_{H^1(\Omega)} = |\Omega|\lambda > 0 \qquad \text{for} \qquad \lambda \neq 0$$

This tells us that $H^1(\Omega)$ is too large of a set for the second condition to hold everywhere. Thus, we want to consider a restriction on $H^1(\Omega)$. Keeping in mind that the average of a constant function is itself, we define

$$\tilde{H} = \{u \in H^1(\Omega) : (u)_\Omega = 0\}$$

equipped with the H^1-norm. To show that $\tilde{H}$ is also a Hilbert space, we will use the fact that closed subsets of Hilbert spaces are also Hilbert spaces. Indeed, let $(u_n) \subset \tilde{H}$ converge to some u. Then

$$\begin{aligned} \left| \int_\Omega u dx \right| &= \left| \int_\Omega u - u_n dx + \int_\Omega u_n \right| \\ &= \left| \int_\Omega u - u_n dx \right| \qquad \text{(since } (u_n)_\Omega = 0) \\ &\leq \sqrt{|\Omega|} \|u - u_n\|_{L^2(\Omega)} \\ &\leq \sqrt{|\Omega|} \|u - u_n\|_{H^1(\Omega)} \\ &\to 0 \end{aligned}$$

so we must have that

$$\int_\Omega u dx = 0$$

or $(u)_\Omega = 0$, so $u \in \tilde{H}$. Thus, $\tilde{H}$ is a Hilbert space. Then we may see that

$$\begin{aligned}
B[u,u] &= \int_\Omega |Du|^2 dx \\
&= \|Du\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}|Du|^2_{L^2(\Omega)} + \frac{1}{2}|Du|^2_{L^2(\Omega)} \\
&\geq \frac{1}{2}|Du|^2_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)} && \text{(Poincare's ineq.)} \\
&\geq \beta\|u\|_{H^1(\Omega)}
\end{aligned}$$

Hence, by Lax-Milgram, we have the existence of a weak solution $\tilde{u} \in \tilde{H}$ such that

$$B[u,v] = \int_\Omega fv dx \qquad \text{for all } v \in \tilde{H}$$

We now want to extend this to all of $H^1(\Omega)$ so let $v \in H^1(\Omega)$. We know that $v - (v)_\Omega \in \tilde{H}$, so

$$\begin{aligned}
B[\tilde{u}, v] &= \int_\Omega D\tilde{u} \cdot Dv dx \\
&= \int_\Omega D\tilde{u} \cdot D(v - (v)_\Omega) dx + \int_\Omega D\tilde{u} \cdot D(v)_\Omega dx \\
&= \int_\Omega D\tilde{u} \cdot D(v - (v)_\Omega) dx \\
&= f(v - (v)_\Omega) dx && (\text{since } (v - (v)_\Omega) \in \tilde{H}) \\
&= \int_\Omega fv dx - (v)_\Omega \int_\Omega f dx \\
&= \int_\Omega fv dx && \text{(by hypothesis)} \\
&= (f, v)
\end{aligned}$$

□

Evans 6.6.10

Assume Ω is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

are $u \equiv C$, for some constant $C \in \mathbb{R}$.

Proof.

(a) Using an energy method, observe that

$$\begin{aligned}
0 = \int_\Omega -u\Delta u dx &= \int_{\partial\Omega} uDu \cdot \nu dS(x) - \int_\Omega -Du \cdot Du dx \\
&= 0 + \int_\Omega |Du|^2 dx && (\text{since } \tfrac{\partial u}{\partial \nu} = 0) \\
&= \int_\Omega |Du|^2 dx
\end{aligned}$$

Thus, we have that $Du = 0$ a.e. in Ω. Since Ω is connected, we use Evans 5.10.11 to conclude that u is constant a.e. in Ω which by smoothness of u, implies that u is constant in Ω.

(b) Suppose u is nonconstant and wlog, assume $u > 0$ somewhere in $\overline{\Omega}$. Then by the smoothness of u, we know that u attains its maximum at some point $x^0 \in \overline{\Omega}$.

- If $x^0 \in \Omega$, then since $Lu = -\Delta u = 0$ and Ω is open, bounded and connected, then the strong maximum principle implies that u must actually be constant.
- If $x^0 \in \partial\Omega$, then since Ω is open and bounded, Ω satisfies the interior ball condition at x^0. Next, we know that u is smooth up to the boundary, so by Hopf's lemma, we must have that
$$\frac{\partial u}{\partial \nu}(x^0) > 0$$
which contradicts that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

Thus, in all cases, we must have that u is constant.

□