Real Analysis Qual. Prep. 2021

Edwin Lin | UCR 2020-21

Contents

1 Theorems and Definitions

σ-Algebras

- **Def.** (Algebra) An algebra of set on $X(\neq \varnothing)$ is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements.
- **Def.** (σ-algebra) A σ-algebra is an algebra that is closed under countable unions.
- **Def.** (Borel σ-algebra) If X is a metric or topological space, then the σ-algebra generated by the family of open sets in X is called the Borel σ-algebra on X and is denoted $\mathcal{B}_X$.
- **Def.** (product σ-algebra) Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of nonempty sets. Let $X = \prod_{\alpha\in A} X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α for each α, the product σ-algebra on X is the σ-algebra generated by the set

$$\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$$

Measures

- **Def.** (Measure) Let X be a nonempty set equipped with σ-algebra, $\mathcal{M}$. A measure on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that
  - (i) $\mu(\varnothing) = 0$
  - (ii) If $(E_j)_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$. (Countable additivity).
- **Thm 1.8** (Properties of measures).

  Let $(X, \mathcal{M}, \mu)$ be a measure space.
  - (a) (Monotonicity) If $E, F \in \mathcal{M}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.
  - (b) (Subadditivity) If $(E_j)_1^\infty \subset \mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$.
  - (c) (Continuity from below) If $(E_j)_1^\infty$ is an increasing sequence in $\mathcal{M}$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.
  - (d) (Continuity from above) If $(E_j)_1^\infty$ is a decreasing sequence in $\mathcal{M}$ and $\mu(E_1) < \infty$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.
- (Types of measures) If $\mu(X) < \infty$ then μ is called a finite measures. If there exists a sequence $(E_j)_1^\infty \subset \mathcal{M}$ such that $X = \bigcup_1^\infty E_j$ and $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, then μ is called a σ-finite measure. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, then μ is called a semifinite measure.
- **Def.** (Complete measure) A measure, μ, whose domain (the σ-alg.) contains all subsets of null-sets is called complete. Null-sets are sets, $N \in \mathcal{M}$ such that $\mu(N) = 0$.
- **Thm 1.9** Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$. The measure $\overline{\mu}$ is called the completion of μ and $\overline{\mathcal{M}}$ is called the completion of $\mathcal{M}$ w.r.t. μ.
- **Def.** (Outer measure) An outer measure on $X(\neq \varnothing)$ is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies
  - (i) $\mu^*(\varnothing) = 0$
  - (ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$
  - (iii) $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$.
- **Def.** (μ^*-measurable sets) A set $A \subseteq X$ is called μ^*-measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ for all } E \subseteq X$$

- **Thm 1.11** (**Caratheodory's Theorem**) If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*- measurable sets forms a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.
- **Def.** (premeasure) If $\mathcal{A}$ is an algebra on X, then a function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a premeasure if $\mu_0(\varnothing) = 0$ and μ_0 is countably additive on disjoint sets.
- (Outer measure induced by premeasure, 1.12)

$$\mu^*(E) = \inf\{\textstyle\sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup_1^\infty A_j\}$$

- **Prop 1.13** If μ_0 is a permeasure on $\mathcal{A}$ and μ^* is an induced outer measure, then $\mu^*|_{\mathcal{A}} = \mu_0$ and every set in $\mathcal{A}$ is μ^*-measurable.

- (Caratheodory's contruction of measures) Start with a premeasure μ_0 on an algebra $\mathcal{A}$, use μ_0 to induce an outer measure μ^*, and then extend μ_0 to a complete measure $\mu = \mu^*|_{\mathcal{M}}$ defined on the σ-algebra, $\mathcal{M}$, of μ^*-measurable sets.

- **Def.** (Lebesgue-Stieltjes measure) Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing, right continuous function. Then there is a unique measure on $\mathcal{B}_{\mathbb{R}}$ (Borel σ-alg. on $\mathbb{R}$) such that the measure of any interval (a, b) is simply its length $b - a$ for all $a, b \in \mathbb{R}$. Caratheodory's contruction may then be applied to extend this measure to a complete measure, denoted μ_F, whose domain, $\mathcal{M}_\mu$, is strictly larger than $\mathcal{B}_{\mathbb{R}}$. This complete measure is called the Lebesgue-Stieltjes measure associated to F and

$$\mu_F(E) = \inf \left\{ \sum_1^\infty [F(b_j) - F(a_j)] : E \subseteq \bigcup_1^\infty (a_j, b_j) \right\}$$

- **Prop 1.20** If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.

- **Def.** (Lebesgue measure) The Lebesgue measure is the complete measure μ_F associated to the function $F(x) = x$. We denote this measure by $m : \mathcal{L} \to [0, \infty]$ where $\mathcal{L}$ denotes the set of Lebesgue measurable sets (m-measurable). Note $\mathcal{B}_{\mathbb{R}} \subset \mathcal{L}$ strictly.

  The most significant properties of the Lebesgue measure are its invariance under translations and simple behavior under dilation.

Measurable Functions

- **Def.** (Measurable functions) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measure spaces. Then a mapping $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable or just measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$. This is similar to the definition of continuous mappings between topological spaces. If $\mathcal{N}$ is a σ-algebra generated by some set $\mathcal{E}$, then we may simply show $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

  For complex-valued functions on X, we say they are measurable if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$-measurable. Such functions have nice closure properties. If $f, g : X \to \mathbb{C}$ are measurable, then so are $f+g$, fg, $\max\{f, g\}$ and $\min\{f, g\}$.

- **Def.** (simple function) A simple function on X is a finite linear combination of characteristic functions of sets in $\mathcal{M}$ with complex coefficients.

$$f = \sum_1^n z_j \chi_{E_j}$$

  where $E_j = f^{-1}(\{z_j\})$ and range$(f) = \{z_j : 1 \leq j \leq n\}$. This is called the standard representation.

- **Thm 2.10** If $f : X \to \mathbb{C}$ is measurable, there is a sequence $(\phi_n)_1^\infty$ of simple functions such that $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$, $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.

- **Prop 2.11** The following are true iff μ is complete:
  (a) If f is measurable and $f = g$ μ-a.e., then g is measurable.
  (b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Integration

- (Integration of nonnegative functions) Define the space $L^+(X)$ to the set of measurable nonnegative functions on X. If ϕ is a simple function in $L^+(X)$ with standard representation $\phi = \sum_1^n a_j \chi_{E_j}$, then define the integral of ϕ w.r.t. μ by

$$\int_X \phi \ d\mu = \sum_1^n a_j \mu(E_j).$$

  and for $A \in \mathcal{M}$, $\int_A \phi \ d\mu = \int_X \phi \chi_A \ d\mu$. Some general properties:
  (a) If $c \geq 0$, $\int c\phi = c \int \phi$.

(b) $\int(\phi+\psi) = \int \phi + \int \psi$

(c) If $\phi \leq \psi$, then $\int \phi \leq \int \psi$.

(d) The map $A \mapsto \int_A d\mu$ is a measure on $\mathcal{M}$.

Now, for any $f \in L^+(X)$, we define its integral by

$$\int f \; d\mu = \sup\left\{\int \phi \; d\mu : 0 \leq \phi \leq f, \; \phi \text{ is simple}\right\}$$

- (Integration of complex-valued functions) For a real-valued function, f, if f^+, f^- are its positive and negative parts and at least one of $\int f^+$ and $\int f^-$ is finite, then we define $\int f = \int f^+ - \int f^-$. If both $\int f^+, \int f^-$ are finite, then we say f is integrable. Note $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

  Next, for a complex-valued function, f, we say that f is integrable on a set E if $\int_E |f| < \infty$ and define

  $$\int f = \int \operatorname{Re} f + i \int \operatorname{Im} f$$

  Note that the space of complex-valued integrable functions is a complex vector space and the integral is a linear functional on it. The space if integrable complex-valued functions on X is denoted $L^1(X)$ or $L^1(\mu)$. Two functions f, g are equivalent in $L^1(X)$ is $f = g$ μ-a.e. $L^1(X)$ is also a metric space with distance $\int |f-g| d\mu$.

- **Thm 2.26** If $f \in L^1(\mu)$ and $\epsilon > 0$, there is an integrable simple function $\phi = \sum_1^n a_j \chi_{E_j}$ such that $\int |f-\phi| d\mu < \epsilon$. That is, the integrable simple functions are dense in L^1 in its metric.

- **Cor 3.6** If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f d\mu| < \epsilon$ whenever $\mu(E) < \delta$.

- **Thm 2.14** (**The Monotone Convergence Theorem**) If $(f_n)_1^\infty \subset L^+$ such that $f_j \leq f_{j+1}$ for all j, and $f = \lim_{n\to\infty}(= \sup_n f_n)$, then $\int f = \lim_{n\to\infty} \int f_n$

- **Prop 2.16** If $f \in L^+$ then $\int f = 0$ iff $f = 0$ a.e.

- **Lemma 2.18** (**Fatou's Lemma**) If $(f_n)_1^\infty$ is any sequence in L^+, then

  $$\int (\liminf f_n) \leq \liminf \int f_n.$$

- **Thm 2.24** (**Dominated Convergence Theorem**) Let $(f_n) \subseteq L^1(X)$ such that

  (a) $f_n \to f$ μ-a.e.

  (b) There exists $g \in L^1$, $g \geq 0$ such that $|f_n| \leq g$ μ-a.e. for all n

  Then $f \in L^1$ and $\int_X f = \lim_{n\to\infty} \int_X f_n$.

- **Thm 2.28**. (Relation between the Lebesgue and Riemann integrals) Let f be a bounded real-valued function on $[a, b]$.

  (a) If f is Riemann integrable, then f is Lebesgue measurable (and hence integrable on $[a, b]$ since it is bounded), and
  $$\int_a^b f(x)dx = \int_{[a,b]} f \; dm.$$

  (b) f is Riemann integrable iff the set of points $x \in [a, b]$ such that f is discontinuous at x has Lebsegue measure zero.

- **Thm 2.26** If $f \in L^1(m)$ then there is a continuous function g that vanishes outside a bounded interval such that $\|f-g\|_1 < \epsilon$

- **Thm 2.27** (Differentiation under the integral sign) Suppose that $f : X \times [a, b] \to \mathbb{C}$ and that $f(\cdot, t) : X \to \mathbb{C}$ is integrable for each $t \in [a, b]$. Let $F(t) = \int_X f(x, t) d\mu(x)$.

  (a) Suppose that there exists $g \in L^1(\mu)$ such that $|f(x, t)| \leq g(x)$ for all x, t. If $\lim_{t\to t_0} f(x, t) = f(x, t_0)$ for every x, then $\lim_{t\to t_0} F(t) = F(t_0)$; in particular, if $f(x, \cdot)$ is continuous for each x, then F is continuous.

  (b) Suppose that $\partial f/\partial t$ exists and there is a $g \in L^1(\mu)$ such that $|(\partial f/\partial t)(x, t)| \leq g(x)$ for all x, t. Then F is differentiable and $F'(x) = \int(\partial f/\partial t)(x, t) d\mu(x)$.

Modes of Convergence

- **Def.** (pointwise convergence) If $(f_n)_1^\infty$ is a sequence of measurable complex-valued functions then $f_n \to f$ pointwise if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. We may also define pointwise μ-a.e. convergence similarly.

- **Def.** (uniform convergence) $(f_n)_1^\infty$ converges to f uniformly if $\|f_n - f\|_\infty = \sup_{x\in X} |f_n(x) - f(x)| \to 0$.

- **Def.** (Convergence in measure) $(f_n)_1^\infty$ converges to f in measure if for every $\epsilon > 0$

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$$

as $n \to \infty$.

- **Def.** (Convergence in L^p space) $(f_n)_1^\infty$ converges to f if $\|f_n - f\|_p = (\int |f_n - f|^p)^{1/p} \to 0$ as $n \to \infty$.

- (Relationships between modes of convergence)

  1. Uniform conv. $\implies$ Pointwise conv. $\implies$ μ-a.e. conv.
  2. If $f_n \to f$ in L^1 then $f_n \to f$ in measure
  3. If $f_n \to f$ in L^1 then there is a subsequence of f_n that converges to f μ-a.e.

- **Thm 2.33** (**Egoroff's Theorem**) Suppose that $\mu(X) < \infty$ and $(f_n)_1^\infty$ and f are all measurable complex-valued functions on X such that $f_n \to f$ μ-a.e. Then for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.

- **Exc 2.44** (**Lusin's Theorem**) If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subseteq [a, b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.

Product Measures

- **Thm 2.37** (**The Fubini-Tonelli Theorem**) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

  (a) (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\begin{aligned}\int f d(\mu \times \nu) &= \int \left[\int f(x, y) d\nu(y)\right] d\mu(x) \\ &= \int \left[\int f(x, y) d\mu(x)\right] d\nu(y)\end{aligned}$$

  (b) (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively and the integral equality of Tonelli's holds as well.

- **Def.** (Lebesgue measure on $\mathbb{R}^n$) The Lebesgue measure on $\mathbb{R}^n$ denoted m^n is the product of Lebesgue measure on $\mathbb{R}$ with itself n times on the n times product space of $\mathcal{B}_{\mathbb{R}}$ or $\mathcal{L}$.

Differentiation of Measures

- **Def.** (signed measure) A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that $\nu(\varnothing) = 0$, ν can only map to either $+\infty$ or $-\infty$ but not both, and if $(E_j)_1^\infty \subset \mathcal{M}$ is disjoint, then $\nu(\bigcup_1^\infty E_j) = \sum_1^\infty \nu(E_j)$ where this sum converges absolutely if $\nu(\bigcup_1^\infty) < \infty$.

  Every signed measure ν can either be represented as the difference between two positive measures $\mu_1 - \mu_2$ or if μ is a measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite, then defining $\nu(E) = \int_E f d\mu$ also produces a signed measure.

- **Thm 3.3** (**The Hahn Decomposition Theorem**) If ν is a signed measure on $(X, \mathcal{M})$, there exists a positive set P and a negative set N for ν such that $P \cup N = X$, $P \cap N = \varnothing$. If another such pair P', N' exists, then $P \triangle P'$ and $N \triangle N'$ are null for ν.

- **Def.** (mutually singular measures) Two signed measures μ, ν on $(X, \mathcal{M})$ are mutually singular if there exists $E, F \in \mathcal{M}$ such that $E \cap F = \varnothing$, $E \cap F = X$ and E is null for μ and F is null for ν. We denote this by $\mu \perp \nu$.

- **Thm 3.4** (**The Jordan Decomposition Theorem**) If ν is a signed measure, there exists unique positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

  Def. (total variation) The total variation of a signed measure ν is $|\nu| = \nu^+ + \nu^-$.

- **Def.** (absoluteley continuous measures) Suppose ν is a signed measure and μ is a positive measure on $(X, \mathcal{M})$. We say that ν is absolutely continuous w.r.t. μ if $\nu(E) = 0$ whenever $\mu(E) = 0$. We denote this by $\nu \ll \mu$.

- **Thm 3.8** (**The Lebesgue-Radon-Nikodym Theorem**) Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. There exists a unique σ-finite signed measure λ, ρ on $(X, \mathcal{M})$ such that

  $$\lambda \perp \mu, \ \ \rho \ll \mu, \ \ \nu = \lambda + \rho.$$

  Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that

  $$d\rho = f d\mu \Leftrightarrow \rho(E) = \int_E f d\mu, \ \ \forall E \in \mathcal{M}$$

  and any two such functions are equal μ-a.e. The decomposition $\nu = \lambda + \rho$ is called the Lebesgue decomposition of ν w.r.t. μ. When $\nu \ll \mu$, we have that $d\nu = f d\mu$ for some f and this f is called the Radon-Nikodym derivatve of ν w.r.t. μ. and is denoted $f = d\nu/d\mu$.

- **Def.** (Hardy-Littlewood maximal function). Let $f \in L^1_{\text{loc}}$, i.e. that f is integrable on any bounded measurable subset of $\mathbb{R}^n$, then

  $$H(f)(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| dy$$

- **Def.** (Lebesgue set) For $f \in L^1_{\text{loc}}$, the Lebesgue set, L_f is defined to be the following:

  $$\left\{ x : \lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy = 0 \right\}$$

- **Def. (The Lebesgue Differentiation Theorem**) Suppose $f \in L^1_{\text{loc}}$. For every x the Lebesgue set of f, in particular, for almost every x, we have

  $$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0$$

  and

  $$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$$

  for every family $\{E_r\}_{r>0}$ that shrinks nicely to x. $\{E_r\}$ shrinks nicely to x if $E_r \subseteq B_r(x)$ for each $r > 0$ and there is some constant α independent of r such that $m(E_r) > \alpha m(B_r(x))$.

- **Thm 3.22** Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$, and let $d\nu = d\lambda + f dm$ be its Lebesgue-Radon-Nikodym representation. Then for m-a.e. $x \in \mathbb{R}^n$,

  $$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$$

  for every family $\{E_r\}_{r>0}$ that shrinks nicely to x. It is particularly useful in application to use balls centered around x.

Differentiation of functions on $\mathbb{R}$

- **Def.** (regular measure) A Borel measure ν on $\mathbb{R}$ will be called regular if $\nu(K) < \infty$ for every compact set K and $\nu(E) = \inf\{\nu(U) : U \text{ open}, E \subseteq U\}$ for every $E \in \mathcal{B}_{\mathbb{R}}$. A signed or complex measure will be called regular if its total variation is regular.

- **Def.** (total variation of a function) Let $F : \mathbb{R} \to \mathbb{C}$. The total variation of F on $[a, b]$ is defined as

  $$T_F([a,b]) = \sup \left\{ \textstyle\sum_1^n |F(x_j) - F(x_{j-1})| : n \in \mathbb{N} \ \ a = x_0 < \cdots < x_n = b \right\}$$

- **Def.** (bounded variation) If $T_F([a, b]) < \infty$ then F is of bounded variation and we denote $F \in BV([a, b])$.

- **Def.** (absolutely continuous function) A function $F : \mathbb{R} \to \mathbb{C}$ is called absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any finite set of disjoint intervals $\{(a_j, b_j)\}_1^N$,

  $$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^N |F(b_j) - F(a_j)| < \epsilon$$

- **Thm 3.35** (The Fundamental Theorem of Calculus for Lebesgue Integrals) If $-\infty < a < b < \infty$ and $F : [a,b] \to \mathbb{C}$, the following are equivalent:

  (a) F is absolutely continuous on $[a,b]$.

  (b) $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a,b], m)$,

  (c) F is differentiable a.e. on $[a,b]$, $F' \in L^1([a,b], m)$, and $F(x) - F(a) = \int_a^x F'(t)dt$.

Point Set Topology

- **Def.** (topology) A topology on X is a family $\mathcal{T}$ of subsets of X that contains $\varnothing$ and X and is closed under arbitrary unions and finite intersections.

- **Def.** (neighborhood base) If $\mathcal{T}$ is a topology on X, a neighborhood base for $\mathcal{T}$ at $x \in X$ is a family $\mathcal{N} \subseteq \mathcal{T}$ such that

  (i) $x \in V$ for all $V \in \mathcal{N}$

  (ii) If $U \in \mathcal{T}$ and $x \in U$, there exists $V \in \mathcal{N}$ such that $x \in V$ and $V \subseteq U$.

  A base for $\mathcal{T}$ is a family $\mathcal{B} \subseteq \mathcal{T}$ that contains a neighborhood base for $\mathcal{T}$ at each $x \in X$.

- **Def.** (first and second countable) A topological space $(X, \mathcal{T})$ is first countable if there is a countable neighborhood base for $\mathcal{T}$ at every point of X. The space is second countable if $\mathcal{T}$ has a countable base.

- **Def.** (separable space) $(X, \mathcal{T})$ is separable if X has a countable dense subset. Every second countable space is separable.

  Def. (Hausdorff space) A space is called Hausdorff if for all $x, y \in X$, $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$.

- **Def.** (weak topology) The weak topology of a topological space $(X, \mathcal{T})$ is the weakest topology (the one with the least open sets) under which every element of X^* is continuous on X.

- **Def.** (weak* topology) The weak* topology is the weakest topology on X^* such that the maps, $T_x(\phi) = \phi(x)$ is continuous on X^* for any $x \in X$. Convergence in the weak* topology is essentially pointwise convergence. That is $f_n \to f$ iff $f_n(x) \to f(x)$ for all $x \in X$.

- **Def.** (nets) To develop a generalization of sequences that work well in arbitrary topological spaces, begin by defining a type of indexed set called a directed set, which is a set $A(\neq \varnothing)$ equipped with a binary relation $\lesssim$ such that

  (i) $\alpha \lesssim \alpha$ for all $\alpha \in A$.

  (ii) if $\alpha \lesssim \beta$ and $\beta \lesssim \gamma$ then $\alpha \lesssim \gamma$.

  (iii) for any $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \lesssim \gamma$ and $\beta \lesssim \gamma$.

  A net in a set X is a mapping $\alpha \mapsto x_\alpha$ from a directed set A into X. Denote such a mapping by $\langle x_\alpha \rangle_{\alpha \in A}$. Let X be a topological space and E a subset of X. A net $\langle x_\alpha \rangle_{\alpha \in A}$ is eventually in E if there exists $\alpha_0 \in A$ such that $x_\alpha \in E$ for $\alpha \gtrsim \alpha_0$. A point $x \in X$ is a limit of $\langle x_\alpha \rangle$ if for every neighborhood U of x, $\langle x_\alpha \rangle$ is eventually in U.

- **Def.** (local compactness) A topological space is locally compact if every $x \in X$ has a neighborhood whose closure is compact.

- **Def.** We call locally compact Hausdorff spaces LCH spaces for short.

- **Def.** The support of a complex-valued function $f : X \to \mathbb{C}$ is defined as

  $$\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$$

  then define the following spaces:

  1. $C(X) = \{f : X \to \mathbb{C} \text{ is continuous}\}$
  2. $BC(X) = \{f \in C(X) : f \text{ bounded}\}$
  3. $C_c(X) = \{f \in C(X) : \text{supp}(f) \text{ compact}\}$
  4. $C_0(X) = \{f \in C(X) : f \text{ vanishes at } \infty\}$

  It may be shown that

  $$C_c(X) \subset C_0(X) \subset BC(X) \subset C(X)$$

- **Lemma 4.32** (**Urysohn's Lemma**) If X is an LCH space and $K \subseteq U \subseteq X$ where K is compact and U is open, there exists $f \in C(X, [0,1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.

- **Prop 4.35** If X is an LCH space, then $C_0(X) = \overline{C_c(X)}$ in $\| \cdot \|_\infty$.

Elements of Functional Analysis

- **Def.** (Banach space) A normed vector space that is complete w.r.t. the norm metric is called a Banach space.

- **Def.** (bounded linear map) A linear map $T : X \to Y$ between two normed vector spaces is called bounded if there exists $C \geq 0$ such that $\|T(x)\|_Y \leq C\|x\|_X$ for all $x \in X$. If T is linear then continuity on X and boundedness on X are equivalent.

- **Def.** (operator norm) Let $L(X, Y)$ be the space of bounded linear maps from $X \to Y$. Then $L(X, Y)$ is a vector space and the function $T \mapsto \|T\|$ is defined by
$$\|T\| = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|T(x)\|_Y$$

- **Def.** (isometry) If $T \in L(X, Y)$, T is called an isometry if $\|T(x)\|_Y = \|x\|_X$. An isometry is injective but not necessarily surjective. It is, however an isomorphism onto its range (i.e. bijective and T^{-1} is bounded).

- **Def.** (dual space) If X is a vector space over $\mathbb{C}$, then a linear map from $X \to \mathbb{C}$ is called a linear functional. If X is a normed vector space then the space $L(X, \mathbb{C})$ of bounded linear functionals on X is called the dual space of X and is denoted by X^*. X^* is a Banach space with its operator norm.

- **Thm 5.6** (**The Hahn-Banach Theorem**) Let X be a real vector space, ρ a sublinear functional on X, $\mathcal{M}$ a subspace of X, and f a complex linear functional on $\mathcal{M}$ such that $|f(x)| \leq \rho(x)$ for all $x \in \mathcal{M}$. Then there exists a complex linear functional F on X such that $|F(x)| \leq \rho(x)$ for all $x \in X$ and $F_{\mathcal{M}} = f$.

- **Thm 5.8** (Consequences of the Hahn-Banach Thm) Let X be a normed vector space.
  (a) If $\mathcal{M}$ is a closed subspace of X and $x \in X \backslash \mathcal{M}$, there exists $f \in X^*$ such that $f(x) \neq 0$ and $f|_{\mathcal{M}} = 0$.
  (b) If $x \neq 0 \in X$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$.
  (c) The bounded linear functions on X separate points.
  (d) If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ be $\hat{x}(f) = f(x)$. Then the map $x \mapsto \hat{x}$ is a linear isometry from X into X^{**} (the dual of X^*).

- **Thm 5.9** (**The Baire Category Theorem**) Let X be a complete metric space.
  (a) If $(U_n)_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X
  (b) X is not a countable union of nowhere dense sets, i.e. not meager.

- **Def.** (meager set) If X is a topological space, a set $E \subseteq X$ is called meager if E is a countable union of nowhere dense sets. A set is called nowhere dense if its closure has empty interior (i.e. no point in it can be contained in an open ball that's contained in the set). Otherwise, E is called residual. Intuitively, nowhere dense sets are naturally very small, so a meager set still has a sense of smallness, but has nicer properties than nowhere dense sets. (σ-ideal).

- **Thm 5.10** (**The Open Mapping Theorem**) Let X, Y both be Banach spaces. If $T \in L(X, Y)$ is surjective, then T is open, i.e. that $T(U)$ is open in Y whenever U is open in X.

- **Thm 5.12** (**The Closed Graph Theorem**) If X, Y are normed vector spaces and T is a linear map from $X \to Y$, define the graph of T to be $\Gamma(T) = \{(x, y) \in X \times Y : y = T(x)\}$. Then T is closed if $\Gamma(T)$ is a closed subspace of $X \times Y$.

  If X, Y are Banach spaces and $T : X \to Y$ is a closed linear map, then T is bounded.

- **Thm 5.13** (**The Uniform Boundedness Principle**) Suppose that X, Y are normed vector spaces and A is a subset of $L(X, Y)$.

(a) If $\sup_{T\in A}\|T(x)\|_Y < \infty$ for all x in some nonmeager subset of X, then $\sup_{T\in A}\|T\| < \infty$

(b) If X is a Banach space and $\sup_{T\in A}\|T(x)\|_Y$ is finite for all $x \in X$, then $\sup_{T\in A}\|T\| < \infty$.

- **Def.** (weak convergence) Let X be a normed vector space. A net $\langle x_\alpha \rangle_{\alpha\in A}$ is said to converge weakly to $x \in X$ iff $f(x_\alpha) \to f(x)$ for all $f \in X^*$.

Hilbert Spaces

- **Def.** (Hilbert Space) Let $\mathcal{H}$ be a complex vector space. An inner product on $\mathcal{H}$ is a map $(x, y) \mapsto \langle x, y\rangle$ from $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that

  (i) $\langle ax + by, z\rangle = a\langle x, z\rangle + b\langle y, z\rangle$ for all $x, y, z \in \mathcal{H}$, and $a, b \in \mathbb{C}$.

  (ii) $\langle y, x\rangle = \overline{\langle x, y\rangle}$

  (iii) $\langle x, x\rangle \in (0, \infty)$ for all nonzero $x \in \mathcal{H}$.

  $\langle \cdot, \cdot\rangle$ induces a norm $\|x\| = \sqrt{\langle x, x\rangle}$ on $\mathcal{H}$ and if $\mathcal{H}$ is complete w.r.t $\|\cdot\|$ then we say $\mathcal{H}$ is a Hilbert space, a special kind of Banach space which generalizes finite Euclidean spaces. Structurally, every Hilbert space looks like some ℓ^2 space (prop 5.30).

- **Thm 5.19** (**The Schwarz Inequality**)

  $|\langle x, y\rangle| \le \|x\|\|y\|$ for all $x, y \in \mathcal{H}$ with equality iff x, y are linearly independent.

- **Thm 5.22** (The Parallelogram Law) For all $x, y \in \mathcal{H}$, $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

- **Thm 5.23** (The Pythagorean Theorem)

  If $(x_j)_1^n \subset \mathcal{H}$ and $x_j \perp x_k$ for $j \neq k$, then

$$\|\sum_1^n x_j\|^2 = \sum_1^n \|x_j\|^2$$

- $L^2(X, \mu)$ is a Hilbert space with inner product $\langle f, g\rangle = \int f\overline{g}d\mu$. An important special case of this is obtained by taking μ to be counting measure on $(X, \mathcal{P}(X))$. Here we denote $L^2(X, \mu)$ be $\ell^2(X, \mu)$ the set of functions $f : X \to \mathbb{C}$ such that $\sum_{x\in X} |f(x)|^2 < \infty$.

- **Thm 5.24** If $\mathcal{M}$ is a closed subspace of $\mathcal{H}$, then $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$; that is , each $x \in \mathcal{H}$ can be uniquely expressed as $x = y + z$ where $y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$. Moreover, y, z are the unique elements of $\mathcal{M}, \mathcal{M}^\perp$ whose distance to x is minimal. Note $\mathcal{M}^\perp$ is called the orthogonal complement of $\mathcal{M}$.

- **Thm 5.25** (**Riesz Representation Theorem for Hilbert Spaces**) If $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ such that $f(x) = \langle x, y\rangle$ for all $x \in \mathcal{H}$.

- **Thm 5.26** (Bessel's Inequality) If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in $\mathcal{H}$, then for any $x \in \mathcal{H}$, $\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \le \|x\|^2$. In particular, the set $\{\alpha : |\langle x, u_\alpha\rangle|^2 \neq 0\}$ is countable.

- **Thm 5.27** (**Parseval's Identity**) If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in $\mathcal{H}$, the following are equivalent:

  (a) If $\langle x, u_\alpha\rangle = 0$ for all $\alpha \in A$, then $x = 0$.

  (b) (Parseval's) $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$ for all $x \in \mathcal{H}$

  (c) For each $x \in \mathcal{H}$, $x = \sum_{\alpha\in A}\langle x, u_\alpha\rangle u_\alpha$, which converges.

- (Some bounded linear operators) Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces.

  1. A unitary map is an invertible (inverse is bounded) map $U : \mathcal{H}_1 \to \mathcal{H}_2$ that preserves inner product. Unitary maps are the true isomorphisms in the category of Hilbert spaces.

  2. Let $\mathcal{H}$ be a Hilbert space and $T \in L(\mathcal{H}, \mathcal{H})$. Then there is a unique $T^* \in L(\mathcal{H}, \mathcal{H})$ called the adjoint of T, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$. Note T is unitary iff T is invertible and $T^{-1} = T^*$.

  3. Let $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace of $\mathcal{H}$ and for $x \in \mathcal{H}$, define $P(x)$ to be the element of $\mathcal{M}$ such that $x - P(x) \in M^\perp$. If defined so, $P \in L(\mathcal{H}, \mathcal{H})$ and $P^* = P$, $P^2 = P$, range$(P) = \mathcal{M}$ and $\ker(P) = \mathcal{M}^\perp$. P is called the orthogonal projection onto $\mathcal{M}$.

L^p Spaces

- **Def.** (L^p space) We define L^p space by the set of measurable functions $f : X \to \mathbb{C}$ such that $\|f\|_p < \infty$ where

$$\|f\|_p = \left[\int_X |f|^p d\mu\right]^{1/p}$$

  If our measure is the counting measure on X then we usually denote L^p space by ℓ^p.

- Two real numbers $p > 1$ and $q > 1$ are called conjugate exponents if $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 1$. then we generally say $q = \infty$ (for norms).

- (**Young's Inequality**) If a, b are nonnegative real numbres and if p, q are conjugate exponents, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

  where equality holds iff $a^p = b^q$.

- **Thm 6.2** (**Holder's Inequality**) Suppose p, q are conjugate exponents. If f, g are measurable functions on X, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

  In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds above iff $\alpha|f|^p = \beta|g|^q$ a.e. for some α, β not both zero.

- **Thm 6.5** (**Minkowski's Inequality**) If $1 \leq p < \infty$ and $f, g \in L^p$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

- **Thm 6.6** For every finite p, L^p is a Banach space.

- **Prop 6.7** For finite p, the set of simple functions $f = \sum_1^n a_j \chi_{E_j}$, where $\mu(E_j) < \infty$ for all j, is dense in L^p.

- **Thm 6.8cde**

  (c) $\|f_n - f\|_\infty \to 0$ iff there exists $E \in \mathcal{M}$ such that $\mu(E^c) = 0$ and $f_n \to f$ uniformly on E.

  (d) L^∞ is a Banach space.

  (e) The simple function are dense in L^∞.

- **Prop 6.10** (Interpolation) If $0 < p < q < r \leq \infty$, then

$$L^p \cap L^r \subseteq L^q \text{ and } \|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-r}$$

  where $\lambda \in (0, 1)$ is defined by

$$\lambda = \frac{q^{-1} - r^{-1}}{p^{-1} - r^{-1}}$$

- **Prop 6.12** (Relationship between L^p spaces) If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supseteq L^q(\mu)$ and

$$\|f\|_p \leq \|f\|_q \mu(X)^{(1/p)-(1/q)}$$

- The most important L^p spaces are L^1 for integrability, L^2 because it is a Hilbert space, and L^∞ because its topology is closely related to that of uniform convergence.

- **Thm 6.15** (**Representation of** $(L^p)^*$) Let p, q be conjugate exponents. If $1 < p < \infty$, then for each $\phi \in (L^p)^*$ there exists $g \in L^q$ such that $\phi(f) = \int fg$ for all $f \in L^p$, and hence L^q is isometrically isomorphic to $(L^p)^*$. The same conclusion holds for $p = 1$ provided μ is σ-finite.

Radon Measures

- **Def.** (regular measure) If μ is a Borel measure on X and E a Borel subset of X. The measure μ is called outer regular on E if

$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$$

  and inner regular on E if

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}$$

  μ is called regular if μ is both outer and inner regular.

- If $f \in C_c(X)$ with $0 \leq f \leq 1$ for all $x \in X$, we write

  1. $K \prec f$ if $f(x) = 1$ for all $x \in K$ where K is compact.
  2. $f \prec V$ if $\text{supp}(f) \subseteq V$ where V is open.

- **Def.** A Borel measure on X is called a Radon measure if

  (i) $\mu(K) < \infty$ for K compact.

  (ii) μ is outer regular for all Borel sets E.

  (iii) μ is inner regular for all open sets E or σ-finite E.

  (iv) μ is complete.

- **Thm 7.2** (**The Riesz Representation Theorem for positive linear functions**) If I is a positive linear functional on $C_c(X)$, there is a unique Radon measure μ on X such that $I(f) = \int f d\mu$ for all $f \in C_c(X)$. Thus, there is a 1-1 correspondence between the set of positive linear functions on $C_c(X)$ and the set of Radon measures on X.

- **Prop 7.9** If μ is a Radon measure on X, $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$.

- Due to the representation theorem, we have 2 ways to determine any Radon measure μ on X:

  1. Either normally by $\mu(E) = \int \chi_E d\mu$, for $E \in \mathcal{M}$
  2. or $\mu(E) = \int_X f d\mu$ for the correct $f \in C_c(X)$.

  The reason is that one can approximate χ_E by $f \in C_c(X)$ when E is nice.

- **Lemma 7.15** If $I \in (C_0(X,\mathbb{R}))^*$, there exists positive functions $I^{\pm} \in (C_0(X,\mathbb{R}))^*$ such that $I = I^+ - I^-$. This is a "Jordan decomposition" for real linear functionals on $C_0(X,\mathbb{R})$.

- **Thm 7.17** (**The Riesz Representation Theorem for** $(C_0(X))^*$) Let X be a LCH space, and for $\mu \in M(X)$ the space of complex Radon measures on X, and $f \in C_0(X)$ let $I_\mu(f) = \int f d\mu$. Then the map $\mu \mapsto I_\mu$ is an isometric isomorphism from $M(X) \to (C_0(X))^*$.

- **Cor 7.18** If X is a compact Hausdorff space, then $(C(X))^*$ is isometrically isomorphic to $M(X)$.

Elements of Fourier Analysis

- $C^\infty(\mathbb{R}^n)$ is the set of infinitely continuously differentiable functions on $\mathbb{R}^n$.

- **Def.** (multi-index notation) We first abbreviate partial derivatives by $\partial_j := \frac{\partial}{\partial x_j}$ in $\mathbb{R}^n$. Now for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we set

$$|\alpha| = \sum_1^n \alpha_j,$$

  and

$$X^\alpha \partial^\beta = \left(\prod_1^n \alpha_j\right) \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$$

- One useful C^∞ space is C_c^∞, the space of compactly supported C^∞ functions. One nontrivial example in this space is

$$\psi(x) = \begin{cases} e^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

- **Def.** (locally convex space and Frechet space) Recall a seminorm is a norm that isn't positive definite (i.e. $\rho(x) = 0$ iff $x = 0$). A family of seminorms $\{\rho_\alpha\}_{\alpha \in A}$ is said to separate points if $\rho_\alpha(x) = 0$ for all $\alpha \in A$ iff $x = 0$.

  A locally convex space is a vector space X with a family of seminorms that separate points. The natural topology on such a space is the weakest topology in which all ρ_α and addition are continuous. This topology may be generated by the set of all open balls w.r.t to each seminorm.

  A locally convect space that is defined by a countable family of seminorms and is complete is called a Frechet space.

- **Def.** (Schwartz space) Schwartz space, $\mathcal{S}$, consists of C^∞ functions which, together with their derivatives, vanish at infinity faster that any power of $|x|$. That is, for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ we define

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

  then

$$\mathcal{S} = \{f \in C^\infty : \|f\|_{(N,\alpha)} < \infty \text{ for all } N, \alpha\}$$

  It is important to note that if $f \in \mathcal{S}$, then $\partial^\alpha f \in L^p$ for all α and all $p \in [1, \infty]$.

- **Prop 8.3** If $f \in C^\infty$, then $f \in \mathcal{S}$ iff $x^\beta \partial^\alpha f$ is bounded for all multi-indices $\alpha, \beta \in \mathbb{R}^n$ iff $\partial^\alpha(x^\beta f)$ is bounded for all multi-indices $\alpha, \beta \in \mathbb{R}^n$. This is a very useful alternative definition for Schwartz functions.

- **Prop 8.2** $\mathcal{S}$ is a Frechet space with the topology defined by the seminorms $\|\cdot\|_{(N,\alpha)}$

- **Def.** (convolution) Let f, g be measurable functions on $\mathbb{R}^n$. The convolution of f and g is the function $f * g$ defined by

$$(f * g)(x) = \int f(x-y)g(y)dy$$

  for all x such that the integral exists.

- **Prop 8.6** Assuming that all integrals in question exist, we have
  - (a) $f * g = g * f$
  - (b) $(f * g) * h = f * (g * h)$
  - (c) For $z \in \mathbb{R}^n$, $\tau_z(f*g) = (\tau_z f)*g = f*(\tau_z g)$ where $\tau_z(f) = f(x-z)$ for all $x \in \mathbb{R}^n$.
  - (d) If A is the closure of $\{x + y : x \in \text{supp}(f), y \in \text{supp}(g)\}$, then $\text{supp}(f*g) \subseteq A$.

- **Prop 8.9** (**Young's Inequality**) Suppose $1 \leq p, q, r \leq \infty$ and $p^{-1} + q^{-1} = r^{-1} + 1$. Then if $f \in L^p$ and $g \in L^q$, then $f * g \in L^r$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

- **Thm 8.15** (Approximate identities) For a function ϕ on $\mathbb{R}^n$ and $t > 0$ we define

$$\phi_t(x) = t^{-n}\phi(t^{-n}x)$$

  If $\phi \in L^1$ and $\int \phi(x)dx = a$ then
  - (a) If $f \in L^p$ $(1 \leq p < \infty)$, then $f * \phi_t \to af$ in the L^p norm as $t \to 0$.
  - (b) If f is bounded and uniformly continuous, then $f * \phi_t \to af$ uniformly as $t \to 0$.
  - (c) If $f \in L^\infty$ and f is continuous on an open set U, then $f * \phi_t \to af$ uniformly on compact subsets of U as $t \to 0$.

- **Prop 8.17** C_c^∞ (and hence also $\mathcal{S}$) is dense in L^p $(1 \leq p < \infty)$ and in C_0.

- **Thm 8.20** Let $E_k(x) = e^{2\pi i k x}$, then $\{E_k : k \in \mathbb{Z}^n\}$ is an orthonormal basis of $L^2(\mathbb{T}^n)$. It is also dense in $C(\mathbb{T}^n)$ which is dense in $L^2(\mathbb{T}^n)$

- **Def.** (Fourier transform on $L^2(\mathbb{T}^n)$) If $f \in L^2(\mathbb{T}^n)$, we define its Fourier transform $\hat{f}$, a function on $\mathbb{Z}^n$, by

$$\mathcal{F}(f)(k) = \hat{f}(k) = \langle f, E_k \rangle = \int f(x)e^{-2\pi i k x}dx$$

  and we call the series

$$\sum_{k \in \mathbb{Z}^n} \hat{f}(k)E_k$$

  the Fourier series of f. The Fourier transform maps $L^2(\mathbb{T}^n)$ onto $\ell^2(\mathbb{Z}^n)$ with $\|\hat{f}\|_2 = \|f\|_2$, and that the Fourier series of f converges to f in the L^2 norm.

- **Thm 8.21** (**The Hausdorff-Young Inequality**) Suppose that $1 \leq p \leq 2$ and q is the conjugate exponent of p. If $f \in L^p(\mathbb{T}^n)$, then $\hat{f} \in \ell^q(\mathbb{Z}^n)$ and $\|\hat{f}\|_q \leq \|f\|_p$.

- **Def.** (Fourier transform on $L^1(\mathbb{R}^n)$) Let $f \in L^1$. Then

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int f(x)e^{-2\pi i \xi x}dx$$

- **Thm 8.22** (Elementary properties of the Fourier transform) Suppose $f, g \in L^1(\mathbb{R}^n)$.
  - (a) $\widehat{(\tau_y f)}(t) = e^{-2\pi i t y}\hat{f}(t)$ and $\tau_y(\hat{f}) = \hat{h}$ where $h(x) = e^{2\pi i y x}f(x)$.
  - (b) If T is an invertible linear tranformation of $\mathbb{R}^n$ and $S = (T^*)^{-1}$ is its inverse transpose, then $\widehat{(f \circ T)} = \hat{f} \circ T$; and if $T(x) = y^{-1}x$, $(y > 0)$, then $\widehat{(f \circ T)}(t) = y^n \hat{f}(yt)$, so that $\widehat{(f_y)}(t) = \hat{f}(yt)$ where $f_y(t) = y^{-n}f(y^{-1}t)$
  - (c) $\widehat{(f * g)} = \hat{f}\hat{g}$.
  - (d) If $x^\alpha f \in L^1$ for $|\alpha| \leq k$ then $\hat{f} \in C^k$ and $\partial^\alpha \hat{f} = [\widehat{(-2\pi i x)^\alpha f}]$.
  - (e) If $f \in C^k$, $\partial^\alpha f \in L^1$ for $|\alpha| \leq k$, and $\delta^\alpha f \in C_0$ for $|\alpha| \leq k-1$, then $\widehat{(\partial^\alpha f)}(t) = (2\pi i t)^\alpha \hat{f}(t)$
  - (f) (**The Riemann-Lebesgue Lemma**) $\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$.

- **Thm 8.26** (**The Fourier Inversion Theorem**) If $f \in L^1$, we define

$$\mathcal{F}^{-1}(f)(x) = \hat{f}(-x) = \int f(\xi) e^{2\pi i \xi x} d\xi.$$

if $\hat{f} \in L^1$ as well, then f agrees almost everywhere with a continuous function f_0, and $\mathcal{F}^{-1}(\hat{f}) = \hat{(\mathcal{F}^{-1}(f))} = f_0$.

References

1. G.B. Folland, Real Analysis: Modern Techniques and Their Applications 2nd Edition.
2. UCR's Real Analysis Qual. Syllabus.
3. University of Houston's Real Analysis Qual. Syllabus.
4. Z. Zhang, UCR Math209C Lecture Notes, 2021.

2 Undergraduate Exercises

2.1 UCR RA Qual 2020

Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then so is f^2. Use only the definition of the derivative.

Solution: Since f is differentiable then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h}$$

exists. Then observe that

$$\begin{aligned}
\lim_{h \to 0} \frac{f^2(x+h) - f^2(x)}{x+h} &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{x+h} \right) (f(x+h) + f(x)) \\
&= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{x+h} \right) \lim_{h \to 0} (f(x+h) + f(x))
\end{aligned}$$

Since both limits above exist by the differentiability of f, then f^2 is also differentiable. ■

2.2 UCR RA Qual 2020

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show, using the $\epsilon - \delta$ definition of continuity, that the composite $f \circ g : \mathbb{R} \to \mathbb{R}$ is continuous.

Solution: Let $c \in \mathbb{R}$ and let $\epsilon > 0$. We want to show $\lim_{x \to c}(f \circ g)(x) = (f \circ g)(c)$. Since f is continuous at $g(c)$, there exists $\delta_1 > 0$ such that $|f(x) - f(g(c))| < \epsilon$ if $|x - g(c)| < \delta_1$. Similarly, since g is continuous at c, then there exists $\delta_2 > 0$ such that $|g(x) - g(c)| < \delta_1$ if $|x - c| < \delta_2$. Then letting $\delta = \min\{\delta_1, \delta_2\}$, if $|x - c| < \delta$, then

$$|g(x) - g(c)| < \delta_1$$

so we have that

$$|f(g(x)) - f(g(c))| < \epsilon$$

Thus, $\lim_{x \to c}(f \circ g)(x) = (f \circ g)(c)$. ■

2.3 UCR RA Qual 2020

Prove or disprove: If $f_n : [0,1] \to \mathbb{R}$ is a sequence of continuous functions and f_n converges uniformly to $f : [0,1] \to \mathbb{R}$, then

$$\int_0^1 f_n dx \to \int_0^1 f dx$$

Tools:

- Theorem 2.24 (Folland): (Dominated Convergence Theorem), Let $(f_n) \subseteq L^1(X)$ such that
  (a) $f_n \to f$ μ-a.e.
  (b) There exists $g \in L^1$, $g \geq 0$ such that $|f_n| \leq g$ μ-a.e. for all n

  Then $f \in L^1$ and $\int_X f = \lim_{n\to\infty} \int_X f_n$.

Solution: In order to invoke the DCT, we will first prove that f is continuous (i.e. the uniform limit of continuous functions is continuous).

Proof. Let $c \in [0,1]$ and $\epsilon > 0$. Then since $f_n \to f$ uniformly, then there exists $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty < \epsilon/3$ for all $n \geq N$. Moreover, since f_N is a continuous function, there exists $\delta > 0$ such that $|f_N(x) - f_N(c)| < \epsilon/3$ if $|x - c| < \delta$. Thus,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c)_f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c)_f(c)| \\ &\leq 2\|f - f_N\| + |f_N(x) - f_N(c)| \\ &< 2(\epsilon/3) + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

□

Thus, f is continuous on [0,1], it must attain its max and min values. Thus, there exists $M \in \mathbb{N}$ such that for every $n \geq M$, $\|f_n - f\| < 1$, so

$$f_n(x) < f(x) + 1, \quad \forall x \in [0,1], 1 \leq n \leq M.$$

Thus, define

$$C = \max(|f+1|) + \max\{\max(|f_n|) : 1 \leq n \leq M\}$$

Then $C \in L^1[0,1]$, $f_n \leq C$ for every n, so by the DCT,

$$\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx. \qquad \blacksquare$$

2.4 UCR RA Qual 2019

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere but whose derivative is not continuous everywhere. Prove it has both these properties.

Solution: Consider the function f given by

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

For $x = 0$, we see that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h \to 0} h \sin(1/h) = 0$$

by the squeeze theorem, so f is differentiable at $x = 0$. Moreover, it is clear that f is differentiable for $x \neq 0$ and

$$f'(x) = \begin{cases} 2x \sin(1/x) + \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Now observe that

$$\lim_{x \to 0^+} \cos(1/x) = \lim_{x \to \infty} \cos(x)$$

which clearly does not exist. Thus, f' is not continuous at $x = 0$. ■

2.5 UCR RA Qual 2020

Prove or disprove: if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then f^2 is also uniformly continuous.

Tools:

- **Def.** (uniform continuity). A function $f : \mathbb{R} \to \mathbb{C}$ is uniformly continuous if for every $\epsilon > 0$, there exist $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$.

Solution: False. Consider the identity function $f(x) = x$ for all $x \in \mathbb{R}$. Then it is clear that f is uniformly continuous since if $|x - y| < \epsilon$ then $|f(x) - f(y)| < \epsilon$. Now suppose that $f^2(x) = x^2$ is uniformly continuous, then there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |x^2 - y^2| < \epsilon$$

However, by choosing $x = \frac{\epsilon}{\delta} + \frac{\delta}{2}$ and $y = \frac{\epsilon}{\delta}$ then $|x - y| < \delta$, but

$$|x^2 - y^2| = |x - y||x + y| = \left(\frac{\delta}{2}\right)\left(\frac{2\epsilon}{\delta} + \frac{\delta}{2}\right) > \epsilon$$

Thus, f^2 is not uniformly continuous. ■

2.6 UCR RA Qual 2019

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at all the irrational numbers and discontinuous at all the rational numbers. Prove it has both these properties.

Solution: Consider the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ fully reduced. } p, q \in \mathbb{Z}. \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

Consider $r \in \mathbb{Q}$, so $r = p/q$ when reduced. Since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, then we know that for any $\delta > 0$, there exists an irrational number y such that $|y - r| < \delta$. But we see that $|f(y) - f(r)| = |0 - 1/q| = |1/q|$ Thus, f is discontinuous at $r \in \mathbb{Q}$.

Now consider $r \in \mathbb{R}\backslash\mathbb{Q}$ and $\epsilon > 0$. Then we first note that there exists some $N \in \mathbb{N}$ such that $1/N > \epsilon$, but $1/(N+1) \leq \epsilon$. Next, for each $1 \leq n \leq N$, we observe that in the interval $[r-1, r+1]$, there are only finitely many $m \in \mathbb{Z}$ such that $m/n \in [r-1, r+1]$. Thus, the number of reduced rational numbers of the form m/n such that $m/n \in [r-1, r+1]$ and $1/n > \epsilon$ must be finite as well. Therefore, define $d = \min\{|m/n - r| : m/n \in [r-1, r+1],\ 1/n > \epsilon\}$. Thus, for any $x \in \mathbb{R}$ such that $|x - r| < d$, if $x \in \mathbb{R}\backslash\mathbb{Q}$ then $|f(x) - f(r)| = 0 < \epsilon$. Otherwise, if $x \in \mathbb{Q}$, then $x = p/q$ when reduced and $|f(x) - f(r)| = |f(x)| = |1/q| < \epsilon$ since it must be that $q > N$. Thus, f is continuous at $r \in \mathbb{R}\backslash\mathbb{Q}$.

2.7 UCR RA Qual 2019

Prove straight from the definition of the Riemann integral that this function $f : [0,1] \to \mathbb{R}$ is Riemann integrable.

$$f(x) = \begin{cases} 0, & \text{if } x \leq 1/2 \\ 1, & \text{if } x > 1/2 \end{cases}$$

Tools:

- (Criterion for Riemann integrability) $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable iff f is bounded and for all $\epsilon > 0$, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ so that $U(f,P) - L(f,P) < \epsilon$ where

  $$U(f,P) = \sum_{j=1}^{n} \left(\sup_{x \in [x_{j-1}, x_j]} f(x) \right) (x_j - x_{j-1})$$

  and

  $$L(f,P) = \sum_{j=1}^{n} \left(\inf_{x \in [x_{j-1}, x_j]} f(x) \right) (x_j - x_{j-1})$$

Solution: Let $\epsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that $1/N < \epsilon$ and consider the partition $P = \{\frac{j}{2N} : 0 \leq j \leq 2N\}$ or $[0,1]$. Then observe that for all $1 \leq j \leq 2N$, $j \neq N+1$,

$$\left(\sup_{x \in [x_{j-1}, x_j]} f(x) - \inf_{x \in [x_{j-1}, x_j]} f(x) \right) = 0$$

since f is constant on $[x_{j-1}, x_j]$. Thus,

$$\begin{aligned}
U(f,P) - L(f,P) &= \sum_{j=1}^{2N} \left(\sup_{x\in[x_{j-1},x_j]} f(x) - \inf_{x\in[x_{j-1},x_j]} f(x) \right)(x_j - x_{j-1}) \\
&= \left(\sup_{x\in\left[\frac{1}{2},\frac{N+1}{2N}\right]} f(x) - \inf_{x\in\left[\frac{1}{2},\frac{N+1}{2N}\right]} f(x) \right)\left(\frac{1}{2N}\right) \\
&= (1-0)\left(\frac{1}{2N}\right) \\
&= \frac{1}{2N} \\
&< \epsilon \quad \blacksquare
\end{aligned}$$

2.8 UCR RA Qeal 2013

Show that $[0,1]$ is uncountable.

Tools:

- **Thm 5.9** (**The Baire Category Theorem**) Let X be a complete metric space.
  (a) If $(U_n)_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X
  (b) X is not a countable union of nowhere dense sets, i.e. not meager.
- A closed subset of a complete metric space is also complete.

  Proof. Suppose (X,ρ) is a complete metric space and $S \subseteq X$ is closed. Let $(x_n)_1^\infty \subset S$ be a Cauchy sequence in S. Then since X is complete, then we know $x_n \to x$ for some $x \in X$, but since S is closed, then $x \in S$. Thus, S is complete. $\square$

Solution: Suppose $[0,1]$ is countable, then

$$[0,1] = \bigcup_{x\in[0,1]} \{x\}$$

It is clear that $\{x\}$ is a closed set with empty interior, hence nowhere dense. Hence $[0,1]$ is the countable union of nowhere dense sets, so by the Baire category theorem, [0,1] is not complete. However this contradicts that $[0,1]$ is a complete metric space since $\mathbb{R}$ is complete and $[0,1]$ is closed. Thus, $[0,1]$ must be uncountable.

3 Part A Exercises

3.1 Folland 1.3

Let $\mathcal{M}$ be a σ-algebra.

(a) $\mathcal{M}$ contains an infinite sequence of disjoint sets.

(b) $\text{card}(\mathcal{M}) \geq \text{card}(\mathbb{R})$.

Tools:

- **Def:** (σ-algebra). Let $X \neq \varnothing$. Then a σ-algebra of sets on X is a nonempty collection $\mathcal{M}$ of subsets of X that is:

  1. closed under countable union
  2. closed under complement

  and hence closed under countable intersection.

Solution:

(a) Since $\mathcal{M}$ is infinite, then there exists $E_1 \in \mathcal{M}$ such that $\varnothing \subset E_1 \subset X$ and the following set

$$A = \{E \cap E_1 : E \in \mathcal{M}\}$$

is infinite. Otherwise, if no such set exists, then for any $\varnothing \subset E_1 \subset X$, A and $B = \{E \cap E_1^c : E \in \mathcal{M}\}$ would both be finite, but then

$$AB = \{F \cup G : F \in A \text{ and } G \in B\}$$

would also be finite, but $\mathcal{M} \subseteq AB$ which contradicts that $\mathcal{M}$ is infinite. Hence, such an E_1 exists and we'll denote $\mathcal{M}_1 = \{E \cap E_1 : E \in \mathcal{M}\}$

<u>Claim:</u> $\mathcal{M}_1$ is a σ-algebra of sets on E_1.

Proof. Let $(B_n \cap E_1)_1^\infty \subseteq \mathcal{M}_1$, then

$$\bigcup_1^\infty B_n \cap E_1 = \left(\bigcup_1^\infty B_n\right) \cap E_1 \in \mathcal{M}_1$$

since $\bigcup_1^\infty B_n \in \mathcal{M}$. Next, let $B \cap E_1 \in \mathcal{M}_1$. Then

$$(B \cap E_1)^c = B^c \cup E_1^c = B^c \cup \varnothing = B^c \cap E_1 \in \mathcal{M}_1$$

Note that the complement above is taken w.r.t. E_1 as our "universe". Thus, $\mathcal{M}_1$ is a σ-algebra on E_1. □

Also, it is clear that E_1^c is disjoint from every set in $\mathcal{M}_1$. Moreover since $\mathcal{M}_1$ is an infinite σ-alg., there exists $E_2 \in \mathcal{M}_1$ such that $\varnothing \subset E_2 \subset E_1$ where

$$\mathcal{M}_2 = \{E \cap E_2 : E \in \mathcal{M}_1\}$$

is again an infinite σ-algebra by the above process and $E_2^c \cap E_1$ is disjoint from every set in $\mathcal{M}_2$ and E_1^c, where the complement is taken w.r.t. X.

Thus, continuing by induction, we generate a disjoint sequence of sets in $\mathcal{M}$:

$$E_1^c, E_2^c \cap E_1, E_3^c \cap E_2, \ldots, E_n^c \cap E_{n-1}, \ldots$$

where the complement is again taken w.r.t. X. ■

(b) Since $\mathcal{M}$ is infinite, then by (a), there exists $(A_j)_1^\infty \subset \mathcal{M}$ where (A_j) is disjoint. Now, consider the set of all increasing sequences of natural numbers,

$$S = \{f : \mathbb{N} \to \mathbb{N} | f \text{ is increasing}\}$$

Claim: S is uncountable.

Proof. Suppose S is only countably infinite, so $S = \{f_1, f_2, \dots\}$. Define the sequence $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} f_1(1) + 1, & n = 1 \\ 1 + \sum_{k=1}^n f_k(k), & n > 1 \end{cases}$$

Then, f is increasing, but $f \neq f_k$ for any $k \in \mathbb{N}$, so $f \notin S$; a contradiction. Thus, S is uncountable. □

Therefore, the collection

$$\mathscr{A} = \left\{ \bigcup_{n\in\mathbb{N}} A_{f(n)} \right\}_{f\in S} \subset \mathcal{M}$$

is an uncountable collection of sets, so $\mathcal{M}$ must at least be uncountable. ■

3.2 Folland 1.4

An algebra $\mathcal{A}$ is a σ-alg. iff $\mathcal{A}$ is closed under countable increasing unions.

Tools:

- **Def:** (algebra). An algebra $\mathcal{A}$ on a set X is a collection of subsets of X that is
  1. closed under finite union
  2. closed under complement

Solution:

($\Rightarrow$) Clear by definition of σ-alg.

($\Leftarrow$) Suppose $\mathcal{A}$ is closed under countable increasing unions and let $(E_j)_1^\infty \subseteq \mathcal{A}$ be an arbitrary sequence of sets in $\mathcal{A}$. Then define $(F_j)_1^\infty$ by

$$F_j = \bigcup_{k=1}^{j} E_k$$

so $(F_j)_1^\infty$ is an increasing sequence of sets in $\mathcal{A}$. Moreover,

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} F_j \in \mathcal{A}.$$

so $\mathcal{A}$ is a σ-alg. ■

3.3 Folland 1.5

$\mathcal{M}(\varepsilon)$ is the union of the σ-algebras generated by F as F ranges over all countable subsets of $\varepsilon \subseteq \mathcal{P}(X)$. (Hint: Show that the latter object is a σ-alg).

Tools:

- Lemma 1.1 (Folland): Let $X \neq \varnothing$. For $\varepsilon, F \subseteq \mathcal{P}(X)$. If $\varepsilon \subseteq \mathcal{M}(F)$, then $\mathcal{M}(\varepsilon) \subseteq \mathcal{M}(F)$.

Solution: Let $S := \{F \subseteq \varepsilon : F \text{ is countable}\}$

($\supseteq$) Since $F \subseteq \varepsilon \subseteq \mathcal{M}(\varepsilon)$, then by Lemma 1.1, $\mathcal{M}(F) \subseteq \mathcal{M}(\varepsilon)$ for all $F \in S$, so

$$\bigcup_{F \in S} \mathcal{M}(F) \subseteq \mathcal{M}(\varepsilon).$$

($\subseteq$) In order to employ a similar strategy, we'll prove the following:
Claim: $\bigcup_{F \in S} \mathcal{M}(F)$ is a σ-alg.

Proof. Let $A \in \bigcup_{F \in S} \mathcal{M}(F)$, then $A \in \mathcal{M}(F)$ for some $F \in S$. Thus, $A^c \in \mathcal{M}(F) \subseteq \bigcup_{F \in S} \mathcal{M}(F)$. Now for $(A_j)_1^\infty \subseteq \bigcup_{F \in S} \mathcal{M}(F)$, we know that there exists $(F_j)_1^\infty \subseteq S$ such that $A_j \in \mathcal{M}(F_j)$. Since each F_j is countable, then $\bigcup_1^\infty F_j$ is countable as well, so $\bigcup_1^\infty F_j \in S$. Hence, by lemma 1.1,

$$A_j \in \mathcal{M}(F_j) \subseteq \mathcal{M}\left(\bigcup_1^\infty F_j\right)$$

for all $j \in \mathbb{N}$. Thus, $\bigcup_1^\infty A_j \in \mathcal{M}\left(\bigcup_1^\infty F_j\right)$, so $\bigcup_1^\infty F_j$ is indeed a σ-alg. □

To now show that $\varepsilon \subseteq \bigcup_{F \in S} \mathcal{M}(F)$, let $E \in \varepsilon$, then $\{E\} \in S$, so

$$E \in \{E\} \subseteq \mathcal{M}(\{E\}) \subseteq \bigcup_{F \in S} \mathcal{M}(F)$$

so by lemma 1.1, $\mathcal{M}(\varepsilon) \subseteq \bigcup_{F \in S} \mathcal{M}(F)$. ■

3.4 Folland 1.6

Prove theorem 1.9 (Folland): Suppose $(X, \mathcal{M}, \mu)$ is a measure space. Let

$$\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$$

and

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subseteq N \in \mathcal{N}\}$$

Then $\overline{\mathcal{M}}$ is a σ-alg. and there is a unique extension $\overline{\mu}$ of μ to a complete measue on $\overline{\mathcal{M}}$.

Tools:

- **Def:** (measure). Let $X \neq \varnothing$ be equipped with a σ-alg. $\mathcal{M}$. A measure on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that
  - (i) $\mu(\varnothing) = 0$
  - (ii) If $(E_j)_1^\infty \subset \mathcal{M}$ is disjoint, then
  $$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$$
- Theorem 1.8ab (Folland): Let $(X, \mathcal{M}, \mu)$ be a measure space.
  - (a) (Monotonicity). If $E, F \in \mathcal{M}$ and $E \subseteq F$ then $\mu(E) \leq \mu(F)$.
  - (b) (Subadditivity). If $(E_j)_1^\infty \subset \mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) \leq \sum_1^\infty \mu(E_j)$.
- **Def:** (null set). $E \in \mathcal{M}$ is a null set if $\mu(E) = 0$. When dealing with multiple measure, we may specify that E is μ-null.
- **Def:** (Complete measure): A measure whose domain (the σ-alg.) contains all subsets of null sets is called complete.

Solution: To show $\overline{\mathcal{M}}$ is a σ-alg., consider $(E_n \cup F_n)_1^\infty \subset \overline{\mathcal{M}}$ where $E_n \in \mathcal{M}$ and $F_n \subseteq N_n$ for some $N_n \in \mathcal{N}$ for each $n \in \mathbb{N}$. Then
$$\bigcup_{n=1}^{\infty} E_n \cup F_n = \bigcup_{n=1}^{\infty} E_N \cup \bigcup_{n=1}^{\infty} F_n \in \overline{\mathcal{M}}$$
since $\bigcup_{n=1}^\infty E_n \in \mathcal{M}$ and $\bigcup_{n=1}^\infty F_n \subseteq \bigcup_{n=1}^\infty N_n \in \mathcal{N}$ and it is clear that a countable union of null sets is still a null set by subadditivity.

Next, for $E \cup F \in \overline{\mathcal{M}}$, $F \subseteq N$, $N \in \mathcal{N}$, we may assume $E \cap N = \varnothing$, otherwise replace F by $F \backslash E$ and N by $N \backslash E$. Then we know that

$$\begin{aligned} E \cup F &= (E \cup N) \cap (N^c \cup F) \\ (E \cup F)^c &= (E \cup N)^c \cup (N^c \cup F)^c \\ &= (E^c \cap N^c) \cup (N \cap F^c) \\ &= (E^c \cap N^c) \cup (N \backslash F) \end{aligned}$$

and we know $(E^c \cap N^c) \in \mathcal{M}$ and $N \backslash F \subseteq N$, so $(E \cup F)^c \in \overline{\mathcal{M}}$, hence $\overline{\mathcal{M}}$ is a σ-alg.

Next, we'll define our extension $\overline{\mu}$ of μ by

$$\overline{\mu}(E \cup F) = \mu(E).$$

This is well-defined since if $E_1 \cup F_1 = E_2 \cup F_2$ where $F_1 \subset N_1 \in \mathcal{N}$ and $F_2 \subset N_2 \in \mathcal{N}$, then we know $E_1 \subseteq E_2 \cup N_2$, so
$$\overline{\mu}(E_1 \cup F_1) = \mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2) = \overline{\mu}(E_2 \cup F_2)$$
and similarly, $\overline{\mu}(E_2 \cup F_2) \leq \overline{\mu}(E_1 \cup F_1)$.

To show that $\overline{\mu}$ on $\overline{\mathcal{M}}$ is complete, simply consider $N \in \mathcal{N}$ and let $F \subset N$. Then $F = \varnothing \cup F \in \overline{\mathcal{M}}$.

Last, suppose $\overline{\nu}$ is also a complete extension of μ over $\overline{\mathcal{M}}$. Let $E \cup F \in \overline{\mathcal{M}}$ where $F \subseteq N \in \mathcal{N}$, and we'll assume $E \cap F = \varnothing$. Then

$$\overline{\nu}(E \cup F) = \overline{\nu}(E) + \overline{\nu}(F) = \mu(E) + \overline{\nu}(F) = \overline{\mu}(E \cup F) + \overline{\nu}(F) = \overline{\mu}(E \cup F).$$

Hence $\overline{\mu}$ is unique. ■

3.5 Folland 1.7

If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_{j=1}^n a_j \mu_j$ is also a measure on $(X, \mathcal{M})$.

Solution: It is clear that $\sum_{j=1}^n a_j \mu_j$ is nonnegative. Next, it is clear that

$$\left(\sum_{j=1}^{\infty} a_j \mu_j\right)(\varnothing) = \sum_{j=1}^{n} a_j \mu_j(\varnothing) = 0.$$

and for a disjoint sequence $(E_j)_1^\infty \subset \mathcal{M}$,

$$\begin{aligned}
\left(\sum_{j=1}^{n} a_j \mu_j\right)\left(\bigcup_{k=1}^{\infty} E_k\right) &= \sum_{j=1}^{\infty} a_j \mu \left(\bigcup_{k=1}^{\infty} E_k\right) \\
&= \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{\infty} \mu_j(E_k)\right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{\infty} a_j \mu_j(E_k) \\
&= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{n} a_j \mu_j\right)(E_k)
\end{aligned}$$

■

3.6 Folland 1.8

If $(X, \mathcal{M}, \mu)$ is a measure space and $(E_j)_1^\infty \subset \mathcal{M}$, then

$$\mu(\liminf E_j) \leq \liminf \mu(E_j).$$

Also, if $\mu\left(\bigcup_1^\infty E_j\right) < \infty$, then

$$\mu(\limsup E_j) \geq \limsup \mu(E_j).$$

Tools:

- **Def:** (set-theoretic limit). Suppose that $(A_n)_1^\infty$ is a sequence of sets. Then

$$\liminf_{n \to \infty} A_n = \bigcup_{n \geq 1} \bigcap_{j \geq n} A_j$$

$$\limsup_{n\to\infty} A_n = \bigcap_{n\geq 1}\bigcup_{j\geq n} A_j$$

- Theorem 1.8cd (Folland): Let $(X, \mathcal{M}, \mu)$ be a measure space.
  (c) (continuity from below): If $(E_j)_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset E_3 \subset \cdots$, then

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

  (d) (continuity from above): If $(E_j)_1^\infty \subset \mathcal{M}$ and $E_1 \supset E_2 \supset \cdots$, and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j)$$

Solution: Let $(E_j)_1^\infty \subset \mathcal{M}$. Since $\bigcap_{k\geq j_1} E_k \subseteq \bigcap_{k\geq j_2} E_k$ for $j_1 \leq j_2$, then by continuity from below,

$$\begin{aligned}
\mu(\liminf E_j) &= \mu\left(\bigcup_{j\geq 1}\bigcap_{k\geq j} E_k\right) \\
&= \lim_{j\to\infty} \mu\left(\bigcap_{k\geq j} E_k\right) \\
&\leq \lim_{j\to\infty} \left(\inf_{k\geq j} \mu(E_k)\right) \\
&= \liminf_{j\to\infty} \mu(E_j)
\end{aligned}$$

Now suppose $\mu\left(\bigcup_{j=1}^\infty E_j\right) < \infty$, then by continuity from above,

$$\begin{aligned}
\mu(\limsup E_j) &= \mu\left(\bigcap_{j\geq 1}\bigcup_{k\geq j} E_k\right) \\
&= \lim_{j\to\infty} \mu\left(\bigcup_{k\geq j} E_k\right) \\
&\geq \lim_{j\to\infty} \left(\sup_{k\geq j} \mu(E_k)\right) \\
&= \limsup_{j\to\infty} \mu(E_j)
\end{aligned}$$

■

3.7 Folland 1.9

If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

Solution: Observe

$$\begin{aligned}\mu(E \cup F) &= \mu((E \backslash F) \cup (E \cap F) + (F \backslash E)) \\ &= \mu(E \backslash F) + \mu(E \cap F) + \mu(F \backslash E) \\ &= \mu(E \cup F) - \mu(F) + \mu(E \cap F) \\ &+ \mu(E \cup F) - \mu(E) \\ &= 2\mu(E \cup F) + \mu(E \cap F) - \mu(E) - \mu(F)\end{aligned}$$

■

3.8 Folland 1.10

Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$. Define $\mu_E(A) = \mu(A \cap E)$ for any $A \in \mathcal{M}$. Then μ_E is a measure on $\mathcal{M}$.

Solution: It's clear that $\mu_E(\varnothing) = \mu(\varnothing \cap E) = 0$. For a disjoint sequence $(A_j)_1^\infty \subset \mathcal{M}$,

$$\begin{aligned}\mu_E\left(\bigcup_{j=1}^{\infty} A_j\right) &= \mu\left(\bigcup_{j=1}^{\infty} A_j \cap E\right) \\ &= \sum_{j=1}^{\infty} \mu(A_j \cap E) \\ &= \sum_{j=1}^{\infty} \mu_E(A_j).\end{aligned}$$

■

3.9 Folland 1.11

Let μ be a finitely additive measure on $(X, \mathcal{M})$. Then

(*i*) μ is a measure iff it is continuous from below.

(*ii*) If $\mu(X) < \infty$, μ is a measure iff it is continuous from above

Solution:

(*i*) Suppose μ is continuous from below. Let $(E_j)_1^\infty \subset \mathcal{M}$ be disjoint, then by finite additivity,

$$\mu\left(\bigcup_{j=1}^{k} E_j\right) = \sum_{j=1}^{k} \mu(E_j)$$

so

$$\lim_{k\to\infty}\mu\left(\bigcup_{j=1}^{k}E_j\right)=\sum_{j=1}^{\infty}\mu(E_j)$$

Let $F_k=\bigcup_{j=1}^{k}E_j$, then $(F_k)_1^\infty\subset\mathcal{M}$ and $F_1\subseteq F_2\subseteq\cdots$, so

$$\begin{aligned}\mu\left(\bigcup_{j=1}^{\infty}E_j\right)=\mu\left(\bigcup_{k=1}^{\infty}F_k\right)&=\lim_{k\to\infty}\mu(F_k)\\&=\lim_{k\to\infty}\mu\left(\bigcup_{j=1}^{k}E_j\right)\\&=\sum_{j=1}^{\infty}\mu(E_j).\end{aligned}$$

(*ii*) Suppose $\mu(X)<\infty$ and μ is continuous from above. Let $(E_j)_1^\infty\subset\mathcal{M}$ be disjoint. Then let $E=\bigcup_{j=1}^{\infty}E_j$ and $F_k=E\backslash\left(\bigcup_{j=1}^{k-1}E_k\right), k\geq 2$ with $F_1=E$.

Claim: $\bigcap_{k=1}^{\infty}F_k=\varnothing$.

Proof. Suppose $\bigcap_{k=1}^{\infty}F_k\neq\varnothing$, so there exists x such that $x\in\bigcap_{k=1}^{\infty}F_k=\bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty}E_j$. Thus, $x\in\bigcup_{j=k}^{\infty}E_j$ for all $k\geq 1$. Then for $k=1$ there exists $N_1\in\mathbb{N}$ such that $x\in E_{n_1}$, but for $k>n_1$, there exists $n_2\neq n_1$ such that $x\in E_{n_2}$, which contradicts that $(E_j)_1^\infty$ is disjoint. □

Now observe that

$$\lim_{k\to\infty}\mu\left(\bigcup_{j=1}^{k}E_j\right)=\sum_{j=1}^{\infty}\mu(E_j)$$

and

$$\mu\left(\bigcup_{j=1}^{\infty}E_j\right)=\mu(E\backslash F_{k+1})=\mu(E)-\mu(F_{k+1})$$

and by continuity from above,

$$\lim_{k\to\infty}\mu(F_{k+1})=\mu\left(\bigcap_{k=1}^{\infty}F_k\right)=0$$

so,

$$\begin{aligned}\sum_{j=1}^{\infty}\mu(E_j)&=\lim_{k\to\infty}\mu\left(\bigcup_{j=1}^{k}E_j\right)\\&=\lim_{k\to\infty}(\mu(E)-\mu(F_{k+1}))\\&=\mu(E)-\lim_{k\to\infty}\mu(F_{k+1})\\&=\mu(E)-\mu\left(\bigcap_{k=1}^{\infty}F_k\right)\\&=\mu(E)\end{aligned}$$

■

3.10 Folland 1.12

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

(a) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.

(b) Say that $E \sim F$ if $\mu(E \triangle F) = 0$. Then $\sim$ is an equivalence relation.

(c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then for $G \in \mathcal{M}$, $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ and hence ρ defines a metric on the space $\mathcal{M} \backslash \sim$ of equivalence classes.

Solution:

(a) Observe,

$$\begin{aligned}
\mu(E) &= \mu((E \triangle F \cup E \cap F) \backslash (F \backslash E)) \\
&= \mu(E \triangle F) + \mu((E \cap F) \backslash (F \backslash E)) \\
&= \mu((E \cap F) \backslash (F \backslash E)) \\
&= \mu(F) - \mu(F \backslash E) \\
&= \mu(F)
\end{aligned}$$

since $F \backslash E \subseteq E \triangle F$.

(b) It is clear that $\sim$ is reflexive and symmetric, so let $E, F, G \in \mathcal{M}$ and suppose that $E \sim F, F \sim G$. Then,

$$\begin{aligned}
\mu(E \backslash G) &= \mu(E \backslash (F \cup G)) + \mu((E \cap F) \backslash G) \\
&\leq \mu(E \backslash F) + \mu(F \backslash G) \\
&\leq \mu(E \triangle F) + \mu(F \triangle G) \\
&= 0.
\end{aligned}$$

Similarly, $\mu(G \backslash E) = 0$, so $E \sim G$ and hence $\sim$ is an equivalence relation.

(c) Since $E \backslash G \subseteq E \backslash F \cup F \backslash G$ and $G \backslash E \subseteq G \backslash F \cup F \backslash E$, then

$$\begin{aligned}
\rho(E, G) &= \mu(E \triangle G) \\
&= \mu(E \backslash G) + \mu(G \backslash E) \\
&= \mu(E \backslash F) + \mu(F \backslash G) + \mu(G \backslash F) + \mu(F \backslash E) \\
&= \mu(E \triangle F) + \mu(F \triangle G) \\
&= \rho(E, F) + \rho(F, G).
\end{aligned}$$

■

3.11 Folland 1.13

Every σ-finite measure is semifinite.

Tools:

- **Def:** (σ-finite measure). A measure μ on $(X, \mathcal{M})$ is called σ-finite if there exists $(E_j)_1^\infty$ such that

$$X = \bigcup_{j=1}^{\infty} E_j$$

  and $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$. Note X need not have infinite measure.

- **Def:** (semifinite measure). A measure μ on $(X, \mathcal{M})$ is called semifinite if for $E \in \mathcal{M}$ with $\mu(E) < \infty$, there exists $F \subset E$ such that

$$0 < \mu(F) < \infty$$

Solution: Let μ be σ-finite. Then $X = \bigcup_{j=1}^{\infty} E_j$ where $\mu(E_j) < \infty$ for all j. Suppose $E \in \mathcal{M}$ and $\mu(E) = \infty$. Then

$$E \cap X = E \cap \bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} E \cap E_j$$

so we have that

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(E \cap E_j) = \infty.$$

Hence, there must exist some $E \cap E_k$ such that $0 < \mu(E \cap E_k) < \infty$, so μ is semifinite. ■

3.12 Example/Counterexample

Find a semifinite measure that is not σ-finite.

Tools:

- (Disjointify) Let $(E_j)_1^\infty$ be a sequence of subsets of X. Then define the sequence $(F_j)_1^\infty$ by

$$\begin{aligned}
F_1 &= E_1 \\
F_2 &= E_2 \backslash E_1 \\
F_3 &= E_3 \backslash (E_1 \cup E_2) \\
&\vdots \\
F_n &= E_n \backslash \left(\bigcup_{j=1}^{n-1} E_j \right) \\
&\vdots
\end{aligned}$$

  Then (F_j) is a disjoint sequence of subsets of X.

Solution Consider $(\mathbb{R}, \mathcal{P}(\mathbb{R}), c)$ where c is defined as the counting measure. We will show c is semifinite on $\mathbb{R}$ but not σ-finite.

c is semifinite since any $E \in \mathcal{P}(\mathbb{R})$ where $c(E) = \infty$ is nonempty, so there is some $\{x\} \subset E$ and $c(\{x\}) = 1$.

Now suppose that c is σ-finite. Then

$$\mathbb{R} = \bigcup_{j=1}^{\infty} E_j$$

where $c(E_j) < \infty$. We may assume that $(E_j)_1^\infty$ is a disjoint sequence, otherwise disjointify it. Since each $c(E_j)$ is finite, we may write $E_j = \{e_{j1}, e_{j2}, \ldots, e_{jc_j}\}$ where $c_j = c(E_j)$. Thus, we may define $f : \bigcup_{j=1}^\infty E_j \to \mathbb{N}$ by

$$f(e_{km_k}) = \left(\sum_{j=1}^{k} c_j\right) + m_k$$

where $0 < m_k \leq c_k$. Suppose $f(e_{km_k}) = f(e_{\ell m_\ell})$, then

$$\left(\sum_{j=1}^{k} c_j\right) + m_k = \left(\sum_{j=1}^{\ell} c_j\right) + m_\ell$$

Now consider the following cases:

1. $(k = \ell, m_k \neq m_\ell)$. This case results in an immediate contradiction.
2. $(k \neq \ell)$. If $k \neq \ell$, then wlog, let $\ell > k$, so

$$\sum_{j=1}^{\ell} c_j = \left(\sum_{j=1}^{\ell} c_j\right) + c_{j+1} + \cdots + c_\ell$$

   Thus, regardless of the value of m_k, m_ℓ, we have a contradiction.

Hence, $k = \ell$, so f is injective. If $n \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that

$$\sum_{j=1}^{k} c_j \leq n < \left(\sum_{j=1}^{k} c_j\right) + c_{k+1}$$

Then, letting $M = k + 1$ and $N = n - \sum_{j=1}^{k} jc_j$, we see that

$$f(e_{MN}) = n$$

Hence, f is a bijection, so $\bigcup_{j=1}^\infty E_j$ is countable infinite which contradicts that $\mathbb{R} = \bigcup_{j=1}^\infty E_j$. Thus, c is not σ-finite. ■

3.13 Folland 1.14

If μ is a semifinite measure and $\mu(E) = \infty$, then for any $c > 0$, there exists $F \subset E$ with $c < \mu(F) < \infty$.

Tools:

- (Technique). When asked to prove that one can "surpass" any positive number, one helpful tip is to find a way to use the supremum of a relevant set and show that it equals infinity. Constructing such a set and invoking the supremum allows one to also construct a sequence and make use of its tools.

Solution: Consider the set

$$F = \{\mu(A) : A \in \mathcal{M}, A \subset E, \mu(A) < \infty\}$$

and let $\sup(F) = s$. Suppose for a contradiction that $s < \infty$ and let $(A_j)_1^\infty \subset \mathcal{M}$, $A_j \subset F$ for all $j \in \mathbb{N}$ be a sequence such that $\mu(A_j) \to s$ as $j \to \infty$. Then let $A = \bigcup_1^\infty A_j$. We know $\mu(A) \geq s$, but if $s < \mu(A) < \infty$ then we contradict that $s = \sup(F)$, so either $\mu(A) = s$ or $\mu(A) = \infty$.

If $\mu(A) = \infty$, then define $(B_j)_0^\infty$ by $B_j = A_j \backslash B_{j-1}$ for $j \geq 1$ and $B_0 = \varnothing$. Then $(B_j)_1^\infty \subset \mathcal{M}$ is a disjointification of $(A_j)_1^\infty$ and

$$\mu\left(\bigcup_{j=1}^\infty B_j\right) = \mu\left(\bigcup_{j=1}^\infty A_j\right) = \mu(A) = \infty$$

so it must be that

$$\mu\left(\bigcup_{j=1}^n B_j\right) = \sum_{j=1}^n \mu(B_j) \to \infty \quad \text{as } n \to \infty$$

so there exists $N \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{j=1}^N B_j\right) = \sum_{j=1}^N \mu(B_j) > s$$

which contradicts that $s = \sup(F)$ since $\bigcup_{j=1}^N B_j \subset E$.

If $\mu(A) = s$, then $\mu(E \backslash A) = \infty$, so there exists $A' \subset E \backslash A$ such that $0 < \mu(A') < \infty$, but $\mu(A \cup A') = \mu(A) + \mu(A') = s + \mu(A') > s$; a contradiction. Hence $s = \infty$. ■

3.14 Folland 1.15

Given a measure μ on $(X, \mathcal{M})$, define μ_0 on $(X, \mathcal{M})$ define μ_0 on $\mathcal{M}$ by

$$\mu_0(E) = \sup\{\mu(F) : F \subset, \mu(F) < \infty\}$$

(a) μ_0 is semifinite. It is called the *semifinite* part of μ.

(b) If μ is semifinite, then $\mu = \mu_0$

(c) There exists a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only the values 0 and ∞ such that $\mu = \mu_0 + \nu$

Solution:

(a) We'll first show that μ_0 is a measure on $\mathcal{M}$. It is clear that $\mu_0(\varnothing) = 0$, so let $(A_j)_1^\infty \subset \mathcal{M}$ be disjoint. Then consider $F \in \mathcal{M}$ such that $F \subset \bigcup_{j=1}^\infty A_j$ with $\mu(F) < \infty$. We know $F = \bigcup_{j=1}^\infty F \cap A_j$ and $\mu(F \cap A_j) < \infty$ for all j, so $\mu(F \cap A_j) \leq \mu_0(A_j)$. Thus,

$$\mu(F) = \sum_{j=1}^\infty \mu(F \cap A_j) \leq \sum_{j=1}^\infty \mu_0(A_j)$$

for any such F, so $\mu_0(\bigcup_{j=1}^\infty) \leq \sum_{j=1}^\infty \mu_0(A_j)$.

Now suppose for a contradiction that $\mu_0(\bigcup_{j=1}^\infty) < \sum_{j=1}^\infty \mu_0(A_j)$ and let $\delta = \sum_{j=1}^\infty \mu_0(A_j) - \mu_0(\bigcup_{j=1}^\infty) > 0$. Then for all $j \in \mathbb{N}$, there exists $F_j \subset A_j$, $\mu(F_j) < \infty$ such that

$$\mu_0(A_j) - \frac{\delta}{2} \cdot 2^{-j} \leq \mu(F_j) \leq \mu_0(A_j).$$

Then since $(A_j)_1^\infty$ is disjoint, then

$$\begin{aligned}\mu\left(\bigcup_{j=1}^\infty F_j\right) = \sum_{j=1}^\infty \mu(F_j) \geq \sum_{j=1}^\infty (\mu_0(A_j) - \frac{\delta}{2} \cdot 2^{-j}) \\ = \sum_{j=1}^\infty \mu_0(A_j) - \frac{\delta}{2} \\ = \mu_0\left(\bigcup_{j=1}^\infty A_j\right) + \frac{\delta}{2}.\end{aligned}$$

Moreover, since $\sum_1^\infty \mu(F_j) > \mu_0(\bigcup_1^\infty A_j)$, then there exists $N \in \mathbb{N}$ such that

$$\sum_{j=1}^N \mu(F_j) = \mu\left(\bigcup_{j=1}^N F_j\right) > \mu_0\left(\bigcup_{j=1}^\infty A_j\right)$$

but $\bigcup_1^N F_j \subset \bigcup_1^\infty A_j$ and since $\mu(F_j) < \infty$ for all j, then $\mu(\bigcup_1^N F_j) < \infty$ as well, so

$$\mu_0\left(\bigcup_{j=1}^\infty A_j\right) < \mu\left(\bigcup_{j=1}^N F_j\right) \leq \mu_0\left(\bigcup_{j=1}^\infty A_j\right)$$

a contradiction. Thus, $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$, so μ_0 is a measure.

To show that μ_0 is semifinite, suppose $\mu_0(E) = \infty$, then for any $c > 0$, by definition of μ_0, there exists some $F \subset E$ such that $c < \mu(F) < \infty$. Then it is clear that $\mu_0(F) = \mu(F)$.

(b) Suppose μ is semifinite. Let $E \in \mathcal{M}$. If $\mu(E) = \infty$, then by Folland 1.14, for every $c > 0$, there exists $F \subset E$ such that $c < \mu(F) < \infty$, so $\mu_0(E) = \infty$. If $\mu(E) < \infty$, then for every $F \subset E$, we know $\mu(F) \leq \mu(E)$. Moreover,$\mu(E) \leq \mu_0(E)$ since $\mu(E) < \infty$. (finite case doesn't need μ semifinite). Hence for all $E \in \mathcal{M}$, $\mu(E) = \mu_0(E)$.

(c) To begin, we first define the notion of a *semifinite set*. Let $E \in \mathcal{M}$ with $\mu(E) = \infty$. Then we say that E is semifinite w.r.t. μ if for every $F \subseteq E$, with $\mu(F) = \infty$, there exists $F' \subset F$ such that $0 < \mu(F') < \infty$.

Using this, we see that if E is semifinite w.r.t. μ, then by Folland 1.14, $\mu(E) = \mu_0(E)$.

Now, define $\nu : \mathcal{M} \to \{0, \infty\}$ by

$$\nu(E) = \begin{cases} 0, & \text{if } \mu(E) < \infty \text{ or } E \text{ is semifinite w.r.t. } \mu \\ \infty, & \text{if } E \text{ is not semifinite w.r.t. } \mu \end{cases}$$

It is clear that $\nu(\varnothing) = 0$, so let $(A_j)_1^\infty \subset \mathcal{M}$ be a disjoint sequence. If $\mu(\bigcup_1^\infty A_j) < \infty$, then $\mu(A_j) < \infty$ for all $j \in \mathbb{N}$, so $\nu(\bigcup_1^\infty A_j) = 0 = \sum_1^\infty \nu(A_j)$.

Next, suppose that $\mu(\bigcup_1^\infty A_j) = \infty$. If $\bigcup_1^\infty A_j$ is semifinite w.r.t. μ, then $\nu(\bigcup_1^\infty A_j) = 0$. Moreover it is clear that A_j is either finite or semifinite w.r.t. μ for all $j \in \mathbb{N}$. Thus, $\sum_1^\infty \nu(A_j) = 0$.

If $\bigcup_1^\infty A_j$ is not semifinite, then $\nu(\bigcup_1^\infty A_j) = \infty$ and there exists $B \subseteq \bigcup_{j=1}^\infty$, $\mu(B) = \infty$ such that for all $B' \subseteq B$, $\mu(B') = 0$ or $\mu(B') = \infty$. Then since $B = \bigcup_1^\infty B \cap A_j$, so $\mu(B \cap A_j) = 0$ or $\mu(B \cap A_j) = \infty$. Since $\mu(B) = \infty$, then there must be some A_k such that $\mu(B \cap A_k) = \infty$, so $\mu(A_k) = \infty$. It is clear that A_k cannot be semifinite since $\mu(B \cap A_k) = \infty$. Therefore,

$$\nu\left(\bigcup_{j=1}^\infty A_j\right) = \infty = \sum_{j=1}^\infty \nu(A_j)$$

So ν is a measure on $\mathcal{M}$ and it is clear that $\mu = \mu_0 + \nu$.

3.15 Folland 1.16

Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subseteq X$ is called *locally measurable* if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ where $\mu(A) < \infty$. Let $\tilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subseteq \tilde{\mathcal{M}}$. If $\mathcal{M} = \tilde{\mathcal{M}}$, then we say that μ is *saturated.*

(a) If μ is σ-finite, then μ is saturated.

(b) $\tilde{\mathcal{M}}$ is a σ-algebra.

(c) Define $\tilde{\mu} : \tilde{\mathcal{M}} \to [0, \infty]$ by

$$\tilde{\mu}(E) = \begin{cases} \mu(E), & \text{if } E \in \mathcal{M} \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\tilde{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$, called the *saturation* of μ.

(d) If μ is complete, so is $\tilde{\mu}$.

(e) Suppose that μ is semifinite. For $E \in \tilde{\mathcal{M}}$, define

$$\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M}, A \subset E\}$$

Then $\underline{\mu}$ is a saturated measure on $\tilde{\mathcal{M}}$ that extends μ.

(f) Let X_1, X_2 be disjoint uncountable sets, $X = X_1 \cup X_2$ and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be the counting measure on $\mathcal{P}(X_1)$ and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\tilde{\mathcal{M}} = \mathcal{P}(X)$, and in the notation of parts (c) and (e), $\tilde{\mu} \neq \underline{\mu}$.

Tools:

- **Def:** (complete measure). A measure is called complete if the σ-algebra it acts upon contains all subsets of null sets.
- **Def:** (counting measure). The counting measure, $\mu_0 : \mathcal{P}(X) \to [0, \infty]$, is defined as

$$\mu_0(E) = \begin{cases} \text{card}(E), & E \text{ finite} \\ \infty, & \text{otherwise} \end{cases}$$

- **Def:** (co-countable). A set E is called co-countable if E^c is countable. The set of co-countable sets forms a σ-algebra.

Solution:

(a) Let μ be σ-finite and let $E \in \tilde{\mathcal{M}}$. Since μ is σ-finite, there exists $(E_j)_1^\infty \subset \mathcal{M}$ such that

$$X = \bigcup_{j=1}^{\infty} E_j, \quad \mu(E_j) < \infty, \qquad \text{for all } j \in \mathbb{N}.$$

Since $E \subset X$ and E is locally measurable, then

$$E = E \cap X = E \cap \left(\bigcup_{j=1}^{\infty} E_j \right) = \bigcup_{j=1}^{\infty} E \cap E_j \in \mathcal{M}$$

so $\tilde{\mathcal{M}} \subseteq \mathcal{M}$ implies $\tilde{\mathcal{M}} = \mathcal{M}$.

(b) Let $E \in \tilde{\mathcal{M}}$. Since E is locally measurable, then for all $A \in \mathcal{M}$ with $\mu(A) < \infty$,

$$(E \cap A) \cup A^c = (E \cup A^c) \cap (A \cup A^c) = (E \cup A^c) \cap X = (E \cup A^c) \in \mathcal{M}$$

so $(E \cup A^c)^c = E^c \cap A \in \mathcal{M}$. Since this holds for all such A, then $E^c \in \tilde{\mathcal{M}}$.
Now let $(E_j)_1^\infty \subset \tilde{\mathcal{M}}$. For any $A \in \mathcal{M}$ where $\mu(A) < \infty$,

$$\left(\bigcup_{j=1}^{\infty} E_j \right) \cap A = \bigcup_{j=1}^{\infty} A \cap E_j \in \mathcal{M}$$

so $\tilde{\mathcal{M}}$ is a σ-alg.

(c) We'll first show $\tilde{\mu}$ is a measure on $\tilde{\mathcal{M}}$. It's clear that $\tilde{\mu}(\varnothing) = 0$. Let $(E_j)_1^\infty \subset \tilde{\mathcal{M}}$ be disjoint. If $E_j \in \mathcal{M}$ for all $j \in \mathbb{N}$, then it's clear that

$$\tilde{\mu}\left(\bigcup_{j=1}^{\infty} E_j \right) = \mu\left(\bigcup_{j=1}^{\infty} E_j \right) = \sum_{j=1}^{\infty} \mu(E_j) = \sum_{j=1}^{\infty} \tilde{\mu}(E_j)$$

Otherwise, there exists $k \in \mathbb{N}$ such that $E_k \notin \mathcal{M}$, so

$$\tilde{\mu}\left(\bigcup_{j=1}^{\infty} E_j \right) = \infty = \sum_{j=1}^{\infty} \tilde{\mu}(E_j)$$

so $\tilde{\mu}$ is a measure on $\tilde{\mathcal{M}}$.

Next, to show $\tilde{\mu}$ is saturated on $\tilde{\mathcal{M}}$, let E be locally measurable on $\tilde{\mathcal{M}}$, i.e. $E \in \tilde{\tilde{\mathcal{M}}}$. Now let $B \in \mathcal{M} \subseteq \tilde{\mathcal{M}}$ with $\mu(B) < \infty$, so

$$\tilde{\mu}(B) = \mu(B) < \infty.$$

Thus, $E \cap B \in \tilde{\mathcal{M}}$, but since $E \cap B$ is locally measurable on $\mathcal{M}$, then $(E \cap B) \cap B = E \cap B \in \mathcal{M}$, so E is locally measurable on M. Thus, $\tilde{\mathcal{M}} = \tilde{\tilde{\mathcal{M}}}$, so $\tilde{\mu}$ is saturated on $\tilde{\mathcal{M}}$.

(d) Let $C \in \tilde{\mathcal{M}}$ be a null set. Then $\tilde{\mu}(N) = 0$, so $N \in \mathcal{M}$. Since μ is complete, then for any $F \subset N$, $F \in \mathcal{M} \subseteq \tilde{\mathcal{M}}$.

(e) To show $\underline{\mu}$ is a measure, it is clear $\underline{\mu}(\varnothing) = 0$. Now let $(E_j)_1^\infty \subset \tilde{\mathcal{M}}$ be disjoint and let $E = \bigcup_1^\infty E_j$.
If $\underline{\mu}(E) < \infty$, then for all $A \subseteq E$, $\mu(A) < \infty$ so by Folland 1.15a, $\underline{\mu}$ is a measure.
If $\underline{\mu}(E) = \infty$, then consider $A \in \mathcal{M}$ with $A \subseteq E$, so $A = \bigcup_1^\infty A \cap E_j$. Then

$$\mu(A) = \sum_{j=1}^\infty \mu(A \cap E_j) \leq \sum_{j=1}^\infty \underline{\mu}(E_j)$$

by definition of $\underline{\mu}$. Taking the sup over all such A, we have

$$\sup_{\substack{A \in \mathcal{M} \\ A \subseteq E}} \mu(A) = \underline{\mu}(E) \leq \sum_{j=1}^\infty \underline{\mu}(E_j).$$

Now suppose by contradiction that $\underline{\mu}(E) < \sum_1^\infty \underline{\mu}(E_j)$. Then for all $j \in \mathbb{N}$, there exists $A_j \in \mathcal{M}$ with $A_j \subseteq E_j$ such that

$$\underline{\mu}(E_j) - \frac{\delta}{2} \cdot 2^{-j} < \mu(A_j) \leq \underline{\mu}(E_j)$$

where $\delta = \sum_1^\infty \underline{\mu}(E_j) - \underline{\mu}(E)$. Since $(E_j)_1^\infty$ is disjoint, then so is $(A_j)_1^\infty$, so

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j) > \sum_{j=1}^\infty \underline{\mu}(E_j) - \frac{\delta}{2}$$

Hence, we see that

$$\underline{\mu}(E) < \sum_{j=1}^\infty \underline{\mu}(E_j) - \frac{\delta}{2} < \mu\left(\bigcup_{j=1}^\infty A_j\right) \leq \underline{\mu}(E)$$

a contradiction. Thus, $\underline{\mu}(E) = \sum_1^\infty \underline{\mu}(E_j)$

Last, to show $\underline{\mu}$ is saturated on $\tilde{\mathcal{M}}$, let E be locally measurable on $\tilde{\mathcal{M}}$. Let $B \in \mathcal{M}$ with $\mu(B) < \infty$. Then $B \in \tilde{\mathcal{M}}$ since $\mu(B) < \infty$ implies $\underline{\mu}(B) < \infty$, so $E \cap B \in \tilde{\mathcal{M}}$, hence $(E \cap B) \cap B = E \cap B \in \mathcal{M}$. Thus, E is locally measurable on $\mathcal{M}$, so $\tilde{\mathcal{M}} = \tilde{\tilde{\mathcal{M}}}$.

(f) First, to show μ is a measure, it is clear that $\mu(\varnothing) = 0$, so let $(A_j)_1^\infty \subset \mathcal{M}$ be disjoint. Then

$$\mu(A) = \mu_0(A \cap X_1) = \mu_0\left(\bigcup_{j=1}^\infty A_j \cap X_1\right) = \sum_{j=1}^\infty \mu_0(A_j \cap X_1) = \sum_{j=1}^\infty \mu(A_j).$$

so μ is a measure on $\mathcal{M}$.

If A is countable, then $E \cap A$ is countable, so $E \cap A \in \mathcal{M}$, so $E \in \tilde{\mathcal{M}}$, hence $\mathcal{P}(X) = \tilde{\mathcal{M}}$. On the other hand, if A is co-countable, then A^c is countable, but since $\mu_0(A \cap X_1) < \infty$, then $A^c \cup (A \cap X_1)$ is also countable, but notice

$$A^c \cup (A \cap X_1) = (A^c \cup A) \cap (A^c \cup X_1) = X \cap (A^c \cup X_1) = A^c \cup X_1$$

but $X_1 \subset A^c \cup X_1$ which contradicts that X_1 is uncountable. Thus, $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$, so E is locally measurable on $\mathcal{M}$. Hence $\mathcal{M} = \mathcal{P}(X)$.

Last, to show $\tilde{\mu} \neq \underline{\mu}$ simply consider $\tilde{\mu}(X_2) = \infty$ since X_2 is neither countable nor co-countable. But $\underline{\mu}(X_2) = 0$ since X_1 and X_2 are disjoint. ■

3.16 Folland 1.17

If μ^* is an outer measure on X and $(A_j)_1^\infty$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\bigcup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$ for any $E \subseteq X$.

Tools:

- **Def:** (outer measure). An outer measure on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that
  - *(i)* $\mu^*(\varnothing) = 0$
  - *(ii)* $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$ (monotonicity)
  - *(iii)* $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$ (subadditivity)
- **Def:** (μ^*-measurable set). A set $A \subseteq X$ is called μ^*-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for any $E \subseteq X$. Whenever $A \subseteq E$, we can think of this definition as saying the outer measure of A is equal to the "inner measure" of A

Solution: Let $E \subseteq X$. Since each A_j is μ^*-measurable, then observe

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) = \mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right) \cap A_1\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right) \cap A_1^c\right)$$
$$= \mu^*(E \cap A_1) + \mu^*\left(E \cap \left(\bigcup_{j=2}^\infty A_j\right)\right)$$

since $(A_j)_1^\infty$ is disjoint. By induction, we see that

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) = \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\left(E \cap \left(\bigcup_{j=n+1}^\infty A_j\right)\right)$$

taking $n \to \infty$ we have

$$\mu^*\left(E \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) = \sum_{j=1}^\infty \mu^*(E \cap A_j) + \lim_{n\to\infty} \mu^*\left(E \cap \left(\bigcup_{j=n+1}^\infty A_j\right)\right) \geq \sum_{j=1}^\infty \mu^*(E \cap A_j)$$

and since $\mu^*(E \cap (\bigcup_1^\infty A_j)) \leq \sum_1^\infty \mu^*(E \cap A_j)$ by definition of the outer measure, then we are done. ■

3.17 UCR RA Qual 2019

State the definition that a set in $\mathbb{R}$ is Lebesgue measurable. Prove that every countable set in $\mathbb{R}$ is Lebesgue measurable.

Tools:

- **Def:** (Lebesgue outer measure). The Lebesgue outer measure is defined as

$$m^*(E) = \inf\left\{\sum_{j=1}^{\infty} b_j - a_j : E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)\right\}$$

  where the restriction of m^* to Lebesgue-measurable sets is called the *Lebesgue measure*. Note the set of Lebesgue-measurable sets is strictly larger than the Borel σ-algebra on $\mathbb{R}$

Solution: Given a set $E \subseteq \mathbb{R}$, E is Lebesgue-measurable if for all $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Let $E \subset \mathbb{R}$ be countable, then we may write $E = \{x_1, x_2, \dots\}$ and the family,

$$\left(x_j - \frac{\epsilon}{2^{j+1}}, x_j + \frac{\epsilon}{2^{j+1}}\right)_{j=0}^{\infty}, \quad \epsilon > 0$$

is a countable cover of E. And since

$$m^*(E) \leq \sum_{j=1}^{\infty} x_j + \frac{\epsilon}{2^{j+1}} - \left(x_j - \frac{\epsilon}{2^{j+1}}\right) = \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+1}} = \frac{\epsilon}{2}$$

so we have that $m^*(E) = 0$. Thus, for any $A \subseteq \mathbb{R}$. Then

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c) = m^*(A \cap E^c) \leq m^*(A)$$

■

3.18 UCR RA Qual 2018

Show that the Dominated Convergence Theorem follows from Fatou's Lemma.

Tools:

- **Def:** (measurable function). Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces (i.e. a set and its σ-alg.). Then a function $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable or just measurable if for every $E \in \mathcal{N}$, $f^{-1}(E) \in \mathcal{M}$. This is a direct analog of continuous functions on topological/metric spaces.
- Proposition 2.11b (Folland): The following implication is valid iff the measure μ is complete:

(b) If f_n is measurable for all $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

- Propsition 2.12 (Folland): Let $(X, \mathcal{M}, \mu)$ be a measure space and $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is a $\overline{\mathcal{M}}$-measurable function on X, then there is a $\mathcal{M}$-measurable function g on X such that $f = g$ $\overline{\mu}$-a.e.

- Lemma 2.18 (Folland): (Fatou's Lemma). If $(f_n)_1^\infty$ is any sequence contained in L^+, (the set of measurable functions from X to $[0, \infty]$), then

$$\int_X \liminf f_n \leq \liminf \int_X f_n$$

- Theorem 2.24 (Folland): (Dominated Convergence Theorem), Let $(f_n)_1^\infty \subseteq L^1(X)$ such that

  (a) $f_n \to f$ μ-a.e.

  (b) There exists $g \in L^1$, $g \geq 0$ such that $|f_n| \leq g$ μ-a.e. for all n

  Then $f \in L^1$ and $\int_X f = \lim_{n\to\infty} \int_X f_n$.

- Recall that $\liminf S = -\limsup(-S)$

Solution: Let $(X, \mathcal{M}, \mu)$ be a measure space, $(X, \overline{\mathcal{M}}, \overline{\mu})$ its completion, and assume the hypothesis of the Dominated Convergence Theorem. Since $f_n \to f$ μ-a.e. then $f_n \to f$ $\overline{\mu}$-a.e. since $\mu(E) = 0 \implies \overline{\mu}(E) = 0$. Thus, by proposition 2.11b, f is $\overline{\mathcal{M}}$-measurable. Thus, by proposition 2.12, there exists a h, $\mathcal{M}$-measurable, such that $f = h$ $\overline{\mu}$-a.e. Let $N \in \overline{\mathcal{M}}$ be the $\overline{\mu}$-null set such that $f \neq h$ on N. Then define $\overline{h}$ by

$$\overline{h}(x) = \begin{cases} h(x), & x \in N^c \\ f(x), & x \in N \end{cases}$$

Then $f = \overline{h}$ for all $x \in X$ and $f = \overline{h}$ is still $\mathcal{M}$-measurable. Moreover, since $f_n \to f$ μ-a.e. then for $\epsilon > 0$ and $x \in X$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for $n \geq N$. Thus,

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| \leq \epsilon + g.$$

Thus, $|f| \leq g$, so

$$\int_X |f| \leq \int g < \infty$$

which means $f \in L^1(X)$. Since f_n and f are complex-valued, by taking their real and imaginary parts, it suffices to assume that f_n and f are real-valued, in which case, we have $g + f_n \geq 0$ and $g - f_n \geq 0$ μ-a.e. Thus, by Fatou's lemma,

$$\int_X g + \int_X f = \int_X (g + \liminf f_n) \leq \liminf \int_X (g + f_n) = \int_X g + \liminf \int_X f_n.$$
$$\int_X g - \int_X f = \int_X (g - \liminf f_n) \leq \liminf \int_X (g - f_n) = \int_X g - \limsup \int_X f_n$$

Thus, we see that

$$\limsup \int_X f_n \leq \int_X f \leq \liminf \int_X f_n$$

so we have that $\lim_{n\to\infty} \int_X f_n = \int_X f$.

3.19 UCR RA Qual 2018

Use Egoroff's theorem to prove the Dominated Convergence theorem for measurable functions on the interval $[0,1]$ with Lebesgue measure.

Tools:

- Theorem 2.33 (Folland). (Egoroff's Theorem). Suppose that $\mu(X) < \infty$ and $f_1, f_2, \ldots$ and f are measurable complex-valued functions on X such that $f_n \to f$ a.e. Then for every $\epsilon > 0$, there exists $E \subseteq X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.
- Corollary 3.6 (Folland). If $f \in L^1(\mu)$, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\int_E f d\mu| < \epsilon$ whenever $\mu(E) < \delta$.

Solution: Assume the hypothesis of the DCT and assume f is a Lebesgue-measurable complex-valued function on $[0,1]$ and let $\epsilon > 0$. Since $f_n \to f$ m-a.e. then for $x \in [0,1]$, $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \epsilon + g$ for n sufficiently large. Thus, $|f| \leq g$, so $f \in L^1(m)$. Moreover, by corr. 3.6, there exists δ such that $|\int_E g \; dm| < \frac{\epsilon}{3}$ when $\mu(E) < \delta$.

Since $m([0,1]) = 1 < \infty$, by Egoroff's theorem, there exists $E \subseteq X$ such that $\mu(E) < \delta$ and $f_n \to f$ uniformly on E^c, hence there exists some $N \in \mathbb{N}$ such that $\|f_n - f\|_\infty < \frac{\epsilon}{3}$ for $n \geq N$. Thus, we see that

$$\begin{aligned}
\left| \int_0^1 f_n - \int_0^1 f \right| &\leq \int_0^1 |f_n - f| \\
&= \int_E |f_n - f| + \int_{E^c} |f_n - f| \\
&\leq 2 \int_E g + \int_{E^c} |f_n - f| \\
&< \frac{2\epsilon}{3} + \|f_n - f\|_\infty \cdot m([0,1]) \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$. ■

3.20 UCR RA Qual 2017

Prove or disprove: if the functions $f_n : [0,1] \to \mathbb{R}$ are continuous and for every $x \in [0,1]$ we have $\lim_{n\to\infty} f_n(x) = 0$, then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

Solution: False, consider the example $f_n(x) = nx(1-x^2)^n$ for all $n \in \mathbb{N}$. We know that $f_n(x)$ is continuous on $[0,1]$ and for $x \in (0,1)$

$$\begin{aligned}\lim_{n\to\infty} nx(1-x^2)^n &= \lim_{n\to\infty} \frac{(1-x^2)^n}{n} \\ &= \lim_{n\to\infty} (1-x^2)^n \ln(1-x^2) \\ &= 0\end{aligned}$$

since $(1-x^2) < 1$ for $x \in (0,1)$. Now observe that

$$\int_0^1 nx(1-x^2)^n dx = \int_0^1 \frac{n}{2} x^n dx = \frac{n}{2(n+1)}$$

which converges to $1/2 \neq 0$.

3.21 UCR RA Qual 2020

Suppose $f_n, f \in L^1([0,1])$. Show that if $f_n \to f$ in measure then

$$\lim_{n\to\infty} \int_0^1 \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} dx$$

Tools:

- **Def.** (convergence in measure). Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n)_1^\infty$ be a sequence of complex-valued measurable functions on X. Then $f_n \to f$ in measure if for any $\epsilon > 0$,
$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0.$$

Solution: Let $\epsilon > 0$. Since $f_n \to f$ in measure then let $E_n = \left\{x \in [0,1] : |f_n(x) - f(x)| \geq \frac{\epsilon}{2}\right\}$. Moreover, since $\mu(E_n) \to 0$ as $n \to \infty$, then there exists some $N \in \mathbb{N}$ such that $\mu(E_n) \leq \frac{\epsilon}{2}$ for $n \geq N$. Thus, for $n \geq N$, observe that

$$\begin{aligned}\left|\int_0^1 \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} dx\right| &= \int_0^1 \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} dx \\ &= \int_{E_n} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} dx + \int_{E_n^c} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} dx \\ &\leq \int_{E_n} dx + \int_{E_n^c} |f_n(x) - f(x)| dx \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \quad \blacksquare\end{aligned}$$

3.22 UCR RA Qual 2020

Let f be a bounded measurable function and g be an integrable function on $\mathbb{R}$. Prove that

$$\lim_{h\to 0} \int_{-\infty}^{\infty} f(x)(g(x+h) - g(x)) dx = 0.$$

Tools:

- The Lebesgue integral is translation invariant.
- **Prop 1.20** (Folland) If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.
- Folland 1.12
- **Thm 2.26** (Folland) If $f \in L^1(\mu)$ and $\epsilon > 0$, there is an integrable simple function $\phi = \sum_1^n a_j \chi_{E_j}$ such that $\int |f - \phi| d\mu < \epsilon$. That is, the integrable simple functions are dense in L^1 in its metric.

Solution: Let $\epsilon > 0$ and let μ be the Lebesgue measure and $\mathcal{L}$ the set of Lebesgue measurable sets. Since f is bounded, there exists some $M \in \mathbb{R}$ such that $|f| \leq M$. Since g is integrable, we know that $g \in L^1$. Also, $g_h := g(x+h) \in L^1$ by the translation invariance of the Lebesgue integral. Thus, by theorem 2.26, there exists some simple integrable function $k = \sum_1^n a_j \chi_{E_j}$ such that $\|k - g\|_1 < \frac{\epsilon}{3M}$. Now, define

$$A + c = \{x + c : x \in A\}$$

where $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Then let $k_h = \sum_1^n a_j \chi_{(E_j - h)}$ so that $\|g_h - k_h\|_1 < \frac{\epsilon}{3M}$ is clear by change of variables.

Since k is integrable, we know $\mu(E_j) < \infty$ for all $1 \leq j \leq n$, so by proposition 1.20, we know that there exists A_j such that $A_j = \bigcup_{k=1}^{m_j} (a_k, b_k)$ and $\mu(E_j \triangle A_j) < \frac{\epsilon}{9n|a_j|M}$. Similarly, by translation, $\mu((E_j - h) \triangle (A_j - h)) < \frac{\epsilon}{9n|a_j|M}$. Let $m := \max\{m_j : 1 \leq j \leq n\}$.

Thus, observe that

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x)(g(x+h) - g(x)) dx \right| &\leq M \int_{\mathbb{R}} |g(x+h) - g(x)| dx \\ &= M \|g_h - g\|_1 \\ &\leq M(\|g_h - k_h\|_1 + \|k_h - k\|_1 + \|k - g\|_1) \\ &< M\left(2\tfrac{\epsilon}{3M} + \|k_h - k\|_1\right) \qquad (*) \end{aligned}$$

Expanding $\|k_h - k\|_1$, we see that

$$\begin{aligned} \|k_h - k\|_1 = \int_{\mathbb{R}} \left| \sum_1^n a_j (\chi_{(E_j - h)} - \chi_{E_j}) \right| dx &= \int_{\mathbb{R}} \left| \sum_1^n a_j \chi_{((E_j - h) \triangle E_j)} \right| dx \\ &\leq \sum_1^n |a_j| \int_{\mathbb{R}} \chi_{((E_j - h) \triangle E_j)} dx \\ &= \sum_1^n |a_j| \mu((E_j - h) \triangle E_j) \qquad (**) \end{aligned}$$

Now by Folland 1.12, we know that $\rho(E, F) = \mu(E \triangle F)$ defines a metric on the set $\mathcal{L}/\sim$ where $E \sim F$ if

$\rho(E, F) = 0$. Thus, by the translation invariance of μ,

$$\begin{aligned}
\mu((E_j - h)\triangle E_j) &\le \mu((E_j - h)\triangle(A_j - h)) + \mu((A_j - h)\triangle A_j) + \mu(A_j \triangle E_j) \\
&< 2\tfrac{\epsilon}{9n|a_j|M} + \mu((A_j - h)\triangle A_j) \\
&= 2\tfrac{\epsilon}{9n|a_j|M} + 2\mu((A_j - h) \cup A_j) - 2\mu(A_j) \\
&= 2\tfrac{\epsilon}{9n|a_j|M} + 2\mu\left(\bigcup_{k=1}^{m_j}(a_k - h, b_k)\right) - 2\mu(A_j) \\
&\le 2\tfrac{\epsilon}{9n|a_j|M} + 2\left(m_j h + \mu(A_j)\right) - 2\mu(A_j) \\
&= 2\tfrac{\epsilon}{9n|a_j|M} + 2m_j h
\end{aligned}$$

Thus, by choosing $|h| < \frac{\epsilon}{18mn|a_j|M}$, we have

$$2\tfrac{\epsilon}{9n|a_j|M} + 2m_j h < 2\tfrac{\epsilon}{9n|a_j|M} + 2m_j \tfrac{\epsilon}{18mn|a_j|M} < \tfrac{\epsilon}{3n|a_j|M}$$

Plugging this back into $(**)$, we have

$$\sum_{j=1}^{n} |a_j|\mu((E_j - h)\triangle E_j) < \sum_{j=1}^{n} |a_j| \tfrac{\epsilon}{3n|a_j|M} = \frac{\epsilon}{3M}$$

which we plug into $(*)$ to see

$$\begin{aligned}
\left| \int_{\mathbb{R}} f(x)(g(x+h) - g(x))dx \right| &\le M \int_{\mathbb{R}} |g(x+h) - g(x)|dx \qquad < M(2\tfrac{\epsilon}{3M} + \|k_h - k\|_1) \\
&< M(2\tfrac{\epsilon}{3M} + \tfrac{\epsilon}{3M}) \\
&= \epsilon
\end{aligned}$$

Hence,

$$\lim_{h \to 0} \int_{-\infty}^{\infty} f(x)(g(x+h) - g(x))dx = 0.$$

3.23 UCR RA Qual 2018

Let f_n be a sequence of measurable real-valued functions on $\mathbb{R}$. Show that

$$A = \{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists}\}$$

is measurable.

Tools:

- Recall that the Lebesgue measurable sets over $\mathbb{R}$ form a σ-algebra.

Solution: First fix $x \in A$. Since $f_n(x)$ converges, then $f_n(x)$ is Cauchy in $\mathbb{R}$. Thus, by the Archimedian property, $f_n(x)$ is Cauchy iff for all $j \in \mathbb{N}$, there exists $N_j \in \mathbb{N}$ such that for all $m, n \geq N_j$,

$$|f_n(x) - f_m(x)| < \frac{1}{j}.$$

In set-theoretic notation, A is equivalent to

$$\bigcap_{j\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{m,n\geq N} \{x \in \mathbb{R} : |f_n(x) - f_m(x)| < \tfrac{1}{j}\}$$

However, we know that $f_n - f_m$ is a measurable function so it must be that

$$\{x \in \mathbb{R} : |f_n(x) - f_m(x)| < \tfrac{1}{j}\} = (f_n - f_m)^{-1}(\tfrac{-1}{j}, \tfrac{1}{j})$$

is measurable as well. Thus, A can be expressed in terms of countable unions and intersections of measurable sets, so A must be measurable.

3.24 Example/Counterexample

In the hypothesis of the Fubini-Tonelli theorem, the 2 measures in question must be σ-finite. Show that the theorem does not hold when either of the measure are not.

Solution: Consider $I = [0, 1]$ and the Borel sets on I. Let μ be the Lebesgue measure on $\mathcal{B}_I$ and ν be the counting measure on 2^I. Then it is clear that ν is not σ-finite while μ is σ-finite.

Let $\Delta = \{(x, x) : x \in I\}$. Then we know that Δ is a closed subset of I^2. Thus, $\Delta \in \mathcal{B}_{I^2}$ and since I is separable, then we know $\mathcal{B}_{I^2} = \mathcal{B}_I \times \mathcal{B}_I \subset \mathcal{B}_I \times 2^I$. Thus, χ_Δ is measurable (via restricting $\chi_{\mathcal{B}_I \times 2^I}$). Moreover, recall that $\Delta_x = \{y \in I : (x, y) \in \Delta\} = \{(x, x)\}$.

$$\begin{aligned}
\int_I \left(\int_I \chi_\Delta(x, y) d\nu(y) \right) d\mu(x) &= \int_I \left(\int_I \chi_{\Delta_x}(y) d\nu(y) \right) d\mu(x) \\
&= \int_I \nu(\{x\}) d\mu(x) \\
&= \nu(\{x\})\mu(I) \\
&= 1
\end{aligned}$$

However, observe on the other hand that

$$\begin{aligned}
\int_I \left(\int_I \chi_\Delta(x, y) d\mu(x) \right) d\nu(y) &= \int_I \left(\int_I \chi_{\Delta^y}(x) d\mu(x) \right) d\nu(y) \\
&= \int_I \mu(\{y\}) d\nu(x) \\
&= \int_I 0 d\nu(x) \\
&= 0
\end{aligned}$$

3.25 Example/Counterexample

Show that Fubini's theorem does not hold for $f \notin L^1(\mu \times \nu)$ for some choice of μ and ν.

Solution: Consider the case where μ, ν are both Lebesgue measure on $\mathcal{B}_I$ where $I = [0,1]$ and consider the function

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

which is continuous except at the origin, hence making it measurable on I^2.

4 Part B Exercises

4.1 Folland 3.1

Prove proposition 3.1: Let ν be a signed measure on $(X, \mathcal{M})$. If $(E_j)_1^\infty \subset \mathcal{M}$ is an increasing sequence then $\nu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$. If $(E_j)_1^\infty \subset \mathcal{M}$ is a decreasing sequence and $\nu(E_1) < \infty$ then $\nu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.

Solution: Let $(E_j)_1^\infty \subset \mathcal{M}$ be an increasing sequence. Then let $F_0 = \varnothing$ and $F_k = E_k \backslash E_{k-1}$ for all $k \in \mathbb{N}$, then $(F_j)_1^\infty \subset \mathcal{M}$ is disjoint, so

$$\nu\left(\bigcup_{j=1}^\infty F_j\right) = \sum_{j=1}^\infty \nu(F_j) = \lim_{n\to\infty} \sum_{j=1}^n \nu(E_j) - \nu(E_{j-1}) = \lim_{n\to\infty} \nu(E_n)$$

Now suppose that $(E_j)_1^\infty$ is decreasing with $\nu(E_1) < \infty$. Then let $F_j = E_1 \backslash E_j$ for $j \in \mathbb{N}$. We see that $\nu(E_1) = \nu(F_j) + \nu(E_j)$ and $\bigcup_1^\infty F_j = E_1 \backslash (\bigcap_1^\infty E_j)$, so

$$\begin{aligned}\nu(E_1) = \nu\left(\bigcap_{j=1}^\infty E_j\right) + \nu\left(\bigcup_{j=1}^\infty F_j\right) &= \nu\left(\bigcap_{j=1}^\infty E_j\right) + \lim_{n\to\infty} \nu(F_n) \\ &= \nu\left(\bigcap_{j=1}^\infty E_j\right) + \lim_{n\to\infty} [\nu(E_1) - \nu(E_n)]\end{aligned}$$

Thus, moving terms around gives the desired result. ■

4.2 UCR RA Qual 2020

Give a function $f : [0,1] \to \mathbb{R}$ that is differentiable at every point (including endpoints, where we use the one-sided derivative) but is not of bounded variation. Prove that it has these properties.

Tools:

- **Def.** (bounded variation). Given a function $f : \mathbb{R} \to \mathbb{C}$, we say that f is of bounded variation on $[a, b]$, denoted $f \in BC([a, b])$ if the total variation of f on $[a, b]$ is finite. Total variation on $[a, b]$ is defined as

$$T_f([a,b]) = \sup \left\{ \sum_{j=1}^{n} |f(x_j) - f(x_{j-1}) : n \in \mathbb{N},\ a = x_0 < x_1 < \cdots < x_n = b \right\}$$

- UCR RA Qual. 2019, undergrad, problem 1. (Solved)

Solution: Consider the following function on $[0, 1]$

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then it is clear that $f(x)$ is differentiable on $[0, 1]$ with derivative

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

To show that f' is not of bounded variation, consider the sequence $x_j = \frac{1}{j\pi}$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned} T_f([0,1]) &\geq \sum_{j=2}^{n} \left| \tfrac{2}{j\pi} \sin(j\pi) - \cos(j\pi) - \tfrac{2}{(j-1)\pi} \sin((j-1)\pi) + \cos((j-1)\pi) \right| + \\ &+ |\tfrac{2}{\pi} \sin(\pi) - \cos(\pi)| \\ &= 1 + \sum_{j=2}^{n} |-2| \\ &= 2n - 1 \end{aligned}$$

Hence, $T_f([0, 1]) \geq n$ for all $n \in \mathbb{N}$, so $f' \notin BV([0, 1])$.

4.3 Folland 3.13

Let $X = [0, 1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure on $[0, 1]$ and μ the counting measure on $\mathcal{M}$.

(a) $m \ll \mu$ but $dm \neq f d\mu$ for any f.

(b) μ has no Lebesgue decomposition w.r.t. m.

Tools:

- **Def.** (mutually singular). Given two signed measure ν and μ on $(X, \mathcal{M})$, we say that ν, μ are mutually singular if there exists sets $E, F \in \mathcal{M}$ such that $E \cap F = 0$, $E \cup F = X$ and E is μ-null and F is ν-null.

- **Def.** (absolutely continuous). Given a signed measure ν and positive measure μ, we say that ν is absolutely continuous with respect to μ, or $\nu \ll \mu$, if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$. Absolute continuity is, in a sense, the antithesis of mutual singularity.
- If $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{M}$ then we denote this relationship by $d\nu = f d\mu$
- Theorem 3.8 (Folland). (The Lebesgue-Radon-Nikodym theorem). Let ν be a σ-finite signed measure and μ a σ-finite positive measure on $(X, \mathcal{M})$. Then there exist unique σ-finite signed measures λ, ρ on $(X, \mathcal{M})$ such that

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \text{and } \nu = \lambda + \rho$$

  Moreover, there is an extended μ-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$, and any two such functions are equal μ-a.e.

  The decomposition $\nu = \lambda + \rho$ where $\lambda \perp \mu$ and $\rho \ll \mu$ is called the Lebesgue decomposition of ν w.r.t. μ. If $\nu \ll \mu$, then an immediate consequence of theorem 3.8 is that $d\nu = f d\mu$ for some f. This result is known as the *Radon-Nikodym* theorem and f is called the *Radon-Nikodym derivative* of ν w.r.t. μ and is commonly denoted $f = d\nu/d\mu$ so that

$$d\nu = \frac{d\nu}{d\mu} d\mu.$$

Solution:

(a) If $E \in \mathcal{M}$ and $\mu(E) = 0$, then $E = \varnothing$. Thus, it is clear that $m(E) = 0$, so $m \ll \mu$. Now suppose by contradiction that there exists some $f : X \to \mathbb{C}$ such that $dm = f d\mu$. Then,

$$m(E) = \int_E f d\mu, \qquad \text{for all } E \in \mathcal{M}$$

Then for any $x \in X$, we know that $\{x\} \in \mathcal{M}$ since singletons are closed. Thus, we see that

$$m(\{x\}) = 0 = \int_{\{x\}} f d\mu = f(x)$$

since we are integrating w.r.t. the counting measure. Thus, f must be the zero function, in which case,

$$m(X) = 1 = \int_X f d\mu = 0.$$

a contradiction. Thus, no such f exists.

(b) Suppose that a Lebesgue decomposition of μ w.r.t. m exists. Then there are unique signed measures λ, ρ such that $\lambda \perp m$, $\rho \ll m$ and $\mu = \lambda + \rho$. Since λ is mutually singular with m, then there exists sets $E, F \in \mathcal{M}$ such that $E \cup F = X$, $E \cap F = \varnothing$ and E is m-null, and F is λ-null.

Observe that F cannot be empty, otherwise $m(X) = m(E \cup F) = m(E) + m(F) = 0 + 0$. Thus, there exists some $x \in F$. Now consider $\{x\} \in \mathcal{M}$, so

$$1 = \mu(\{x\}) = (\lambda + \rho)(\{x\}) \leq \lambda(F) + \rho(F) = 0$$

a contradiction. ■

4.4 Folland 3.2

If ν is a signed measure, E is ν-null iff $|\nu|(E) = 0$. Also if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Tools:

- **Def.** (μ-null). A set $E \in \mathcal{M}$ is null w.r.t. a measure μ or μ-null if for every $F \subseteq E$, $\mu(F) = 0$.
- Theorem 3.3 (Folland). (Hahn-Decomposition theorem). If ν is a signed measure on $(X, \mathcal{M})$ then there exists a positive set P and negative set N such that $P \cup N = X$, $P \cap N = \varnothing$. If P', N' is another such pair, then $P \triangle P'$ and $N \triangle N'$ are both ν-null.
- Theorem 3.4 (Folland). (Jordan-Decomposition theorem). If ν is a signed measure, then there exists unique positive measures ν^+, ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Solution: Suppose that E is ν-null. If P, N is a Hahn-decomposition of ν, then $E \cap P$ and $E \cap N$ are both also ν-null. So it is clear that $\nu(E \cap P) = \nu^+(E) = 0$ and $\nu(E \cap N) = \nu^-(E) = 0$, so $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$.

Next, if $\nu \perp \mu$ then there exists sets E, F such that $E \cup F = X$, $E \cap F = \varnothing$ and E is ν-null, and F is μ-null. Then for every $\tilde{E} \subseteq E$, $|\nu|(\tilde{E}) = 0$, so E is $|\nu|$-null, so $|\nu| \perp \mu$.

If $|\nu| \perp \mu$ then since ν^+ and ν^- are bounded by $|\nu|$ then we know that $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Last, if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$ then there are sets E^+, F^+, E^-, F^- such that $E^+ \cup F^+ = X = E^- \cup F^-$ and $E^+ \cap F^+ = \varnothing = E^- \cap F^-$, with E^+ ν^+-null, E^- ν^--null and F^+, F^- μ-null. Then consider $E = E^+ \cap E^-$ and $F = F^+ \cup F^-$, so that E is ν-null since $\nu = \nu^+ - \nu^-$ and F is μ-null. Moreover, it is clear that $E \cup F = X$ and $E \cap F = \varnothing$, so $\nu \perp \mu$. ■

4.5 UCR RA Qual 2016

Show that every weakly convergent sequence in a Banach space X is bounded with respect to the norm of the Banach space.

Tools:

- **Def.** (weak convergence) If X is a normed vector space, then we say that a sequence $(x_n)_1^\infty$ converges weakly to $x \in X$ if $f(x_n) \to f(x)$ for all $f \in X^*$, its dual space.
- Theorem 5.8d (Folland). If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ be $\hat{x}(f) = f(x)$. Then the map $x \mapsto \hat{x}$ is a linear isometry from X into X^{**} (the dual of X^*).
- Theorem 5.13 (Folland). (The Uniform Boundedness Principle) Suppose that X, Y are normed vector spaces and A is a subset of $L(X, Y)$.

(a) If $\sup_{T\in A} \|T(x)\|_Y < \infty$ for all x in some nonmeager subset of X, then $\sup_{T\in A} \|T\| < \infty$

(b) If X is a Banach space and $\sup_{T\in A} \|T(x)\|_Y < \infty$ for all $x \in X$, then $\sup_{T\in A} \|T\| < \infty$.

- If X is a normed vector space then X^*, its dual space, is a Banach space with the operator norm.

Solution: Let $(x_n)_1^\infty$ be a weakly convergent sequence in X to some $x \in X$. We want to show that $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$. By the definition of weak convergence, we know that for every bounded linear functional $f \in X^*$, $f(x_n) \to f(x)$ in $\mathbb{C}$. By theorem 5.8d, we know that we may define a map $x \mapsto \hat{x}$ where $\hat{x} : X^* \to \mathbb{C}$ with $\hat{x}(f) = f(x)$ and such a mapping is a linear isometry from X to X^{**}. Thus,

$$\sup_{n\in\mathbb{N}} \|x\|_X = \sup_{n\in\mathbb{N}} \|\hat{x}_n\| = \sup_{n\in\mathbb{N}} \left(\sup_{\substack{f\in X^* \\ \|f\|=1}} |\hat{x}_n(f)| \right)$$

and since $f(x_n) \to f(x)$ in $\mathbb{C}$, we know that

$$\sup_{n\in\mathbb{N}} \|\hat{x}_n(f)\| = \sup_{n\in\mathbb{N}} |f(x_n)| < \infty$$

Thus, by the Uniform Boundedness Principle, we know that

$$\sup_{n\in\mathbb{N}} \|\hat{x}_n(f)\| < \infty \implies \sup_{n\in\mathbb{N}} \|\hat{x}\|_X = \sup_{n\in\mathbb{N}} \|x\|_X < \infty \quad \blacksquare$$

4.6 UCR 209B 2021 Final

Let $X = [0, 2\pi]$ equipped with the Lebesgue measure.

(a) Let $f_n(x) = \sin^3(nx)$. Prove that f_n converges to 0 weakly in L^2. You may assume the Riemann Lebesgue lemma.

(b) Prove that f_n does not converge to 0 m-a.e.

Tools:

- Theorem 7.2 (Folland). (The Riesz Representation Theorem for Hilbert Spaces on $\mathbb{R}$) Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For every continuous linear functions $\varphi \in H^*$, there exists a unique $f_\varphi \in H$ such that
$$\varphi(x) = \langle x, f_\varphi \rangle$$
for all $x \in H$. Note that if the underlying field is $\mathbb{R}$ then H is isometrically isomorphic to H^*.

- Theorem 8.22 (Folland). (Riemann-Lebesgue Lemma) $\mathcal{F}(L^1(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$. Note $\mathcal{F}$ is the Fourier transform.

Solution: We first observe that $L^2[0,2\pi]$ is a Hilbert space. Now, let $T \in (L^2[0,2\pi])^*$. In order to show that $T(f_n) \to 0$, we apply the Riesz Representation theorem for Hilbert spaces, so there exists $g_T \in L^2[0,2\pi]$ such that $T(f) = \langle f, g_T\rangle$ for all $f \in L^2[0,2\pi]$. Thus, it suffices to show that $\langle f_n, g_T\rangle \to 0$.

Next, notice that

$$f_n(x) = \sin^3(nx) = \left(\frac{e^{inx} - e^{-inx}}{2i}\right)^3 = -\frac{1}{8i}(e^{3inx} - 3e^{inx} + 3e^{-inx} - e^{-3inx})$$

Thus,

$$\langle f_n, g_T\rangle = -\frac{1}{8i}\int_0^{2\pi}(e^{3inx} - 3e^{inx} + 3e^{-inx} - e^{-3inx})\overline{g_T}\ dx$$

which are some of the Fourier coefficients of $\overline{g_T}$. Moreover, by Holder's inequality,

$$\int_0^{2\pi} |g_T|dx \le \|g_T\|_2\|1\|_2 = \|g_T\|_2\ m([0,2\pi]) < \infty.$$

Thus, $L^2[0,2\pi] \subset L^1[0,2\pi]$, so by the Riemann Lebesgue lemma, we know that $\langle f_n, g_T\rangle \to 0$.

To show $f_n \not\to 0$ m-a.e., assume by contradiction that $f_n \to 0$, then we know $f_n^2 \to 0$ as well. It is clear that $f_n^2 \in L^1[0,2\pi]$ for all $n \in \mathbb{N}$ and that f_n^2 is dominated by 1. Thus, the Dominated Convergence theorem implies that $\|f_n\|_2^2 \to 0$ as $n \to \infty$. However, observe that

$$\|f_n\|_2^2 = \int_0^{2\pi} \sin^6(nx)dx = \frac{5\pi}{8} \neq 0$$

a contradiction. Thus, $f_n \not\to 0$ a.e.

4.7 UCR RA Qual 2016

Let g be a Lipschitz function on $[0,1]$ and f be an absolutely continuous function from $[0,1] \to [0,1]$. Prove that the composite $g \circ f$ is also absolutely continuous.

Tools:

- **Def.** (absolutely continuous function) Let $f : \mathbb{R} \to \mathbb{C}$. Then f is absolutely continuous on $\mathbb{R}$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that for any finite family of disjoint intervals $\{(a_j, b_j\}_{j=1}^N$, $N \in \mathbb{N}$, then
$$\sum_1^N (b_j - a_j) < \delta \implies \sum_1^N |f(b_j) - f(a_j)| < \epsilon$$

Solution: Let $\epsilon > 0$. Let $C \in \mathbb{R}$ be the Lipschitz constant of g. Since f is absolutely continuous on $[0,1]$, then there exists δ such that

$$\sum_1^N |f(b_j) - f(a_j)| < \frac{\epsilon}{C}$$

whenever $\sum_1^N (b_j - a_j) < \delta$ for any finite family of disjoint intervals $\{(a_j, b_j)\}_1^N$. Then since g is Lipschitz,

$$\begin{aligned}\sum_1^N |(g \circ f)(b_j) - (g \circ f)(a_j)| &\leq \sum_1^N C|f(b_j) - f(a_j)| \\ &= C \sum_1^N |f(b_j) - f(a_j)| \\ &< C\frac{\epsilon}{C} \\ &= \epsilon\end{aligned}$$

4.8 UCR RA Qual 2016

Let F be the linear functional on $C[-1,1]$ defined by

$$F(x) = \int_{-1}^{0} x(t)dt - \int_0^1 x(t)dt, \quad \forall x \in C[-1,1]$$

Prove that $\|F\| = 2$.

Solution: First observe that

$$\begin{aligned}\|F\| &= \sup_{\substack{x \in C[-1,1] \\ \|x\|_\infty = 1}} |F(x)| \\ &= \sup_{\substack{x \in C[-1,1] \\ \|x\|_\infty = 1}} \left| \int_{-1}^0 x - \int_0^1 \right| \\ &\leq \sup_{\substack{x \in C[-1,1] \\ \|x\|_\infty = 1}} \left(\left| \int_{-1}^0 x \right| + \left| \int_0^1 x \right| \right) \\ &\leq \sup_{\substack{x \in C[-1,1] \\ \|x\|_\infty = 1}} \left(\int_{-1}^1 |x| \right) \\ &\leq \|x\|_\infty \int_{-1}^1 dt \\ &= 2.\end{aligned}$$

Now suppose by contradiction that $\|F\| < 2$ and let $\delta = 2 - \|F\|$. Then there exists $N \in \mathbb{N}$ such that $1/N < \delta$, so consider the function $x_N(t) \in C[-1,1]$ where

$$x_N(t) = \begin{cases} 1, & 1/N < t \leq 1 \\ Nt, & |t| \leq 1/N \\ -1, & -1 \leq t < -1/N \end{cases}$$

It is clear that $\|x_N\|_\infty = 1$. Next, observe that

$$|F(x_N)| = \left| \int_{-1}^0 x_N - \int_0^1 x_N \right| = 2 \int_0^1 x_N = 2 \left(\int_0^{1/N} Ntdt + (1 - 1/N) \right)$$

and $\int_0^{1/N} Ntdt = 1/2N$, so

$$|F(x_N)| = 2(1/2N + 1 - 1/N) = 2 - 1/N > 2 - \delta > \|F\|$$

a contradiction. Thus, $\|F\| = 2$.

4.9 Folland 3.22

If $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exists $C, R > 0$ such that $Hf(x) > C|x|^{-n}$ for $|x| > R$.

Tools:

- **Def.** (Hardy-Littlewood maximal function). Let $f \in L^1_{\text{loc}}$, i.e. that f is integrable on any bounded measurable subset of $\mathbb{R}^n$, then

$$H(f)(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)|dy$$

- Prop 2.16 (Folland) if $f \in L^+$, then $\int f = 0$ iff $f = 0$ a.e.

Solution: We first observe that since $f \neq 0$ and $f \in L^1(\mathbb{R}^n)$, then for some $R > 0$,

$$0 < \int_{B_R(0)} |f|dm < \infty.$$

Otherwise, if no such R exists, then by proposition 2.16, $|f| = 0$ a.e. so $f = 0$. Now for $x \in \mathbb{R}^n$ with $|x| > R$, we know that $B_{2|x|}(x) \supset B_R(0)$. Moreover, by the translation invariance of the Lebesgue measure, we know $m(B_{2|x|}(x)) = m(B_{2|x|}(0)) = \alpha_n |x|^n$ for some $\alpha \in \mathbb{R}$. Thus,

$$\begin{aligned} Hf(x) &= \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)|dy \\ &\geq \frac{1}{m(B_{2|x|}(x))} \int_{B_{2|x|}(x)} |f|dm \\ &= \frac{1}{\alpha_n} |x|^{-n} \int_{B_{2|x|}(x)} |f|dm \\ &\geq \frac{1}{\alpha_n} |x|^{-n} \int_{B_R(0)} |f|dm \end{aligned}$$

Thus, let $C = \frac{1}{2\alpha_n} \int_{B_R(0)} |f|dm$ so that we have

$$Hf(x) > C|x|^{-n}, \quad \text{for all } |x| > R.$$

4.10 UCR 209B 2021 Final

Show that $L^2[0,1]$ is a meager subset of $L^1[0,1]$.

Tools:

- **Def.** (meager set) If X is a topological space, a set $E \subseteq X$ is called meager if E is a countable union of nowhere dense sets. A set is called nowhere dense if its closure has empty interior (i.e. no point in it can be contained in an open ball that's contained in the set). Otherwise, E is called residual. Intuitively, nowhere dense sets are naturally very small, so a meager set still has a sense of smallness, but has nicer properties than nowhere dense sets. (σ-ideal).
- If $f_n \to f$ in L^1 then there is a subsequence of f_n that converges to f μ-a.e.
- **Thm 6.6** For every finite p, L^p is a Banach space.
- **Lemma 2.18** (**Fatou's Lemma**) If $(f_n)_1^\infty$ is any sequence in L^+, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$

- For $f_n, f : \mathbb{R} \to \mathbb{C}$, $f_n \to f$ pointwise implies $f_n^2 \to f^2$.

  Proof. Let $f \geq 0$. Then since $f_n \to f$, for $x \in \mathbb{R}$, there exists $N_1 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < f(x)+1$ for $n \geq N_1$, so $f_n(x) \in (-1, 2f(x)+1)$. Moreover, there is $N_2 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/|3f(x)+1|$, so for $n \geq N > \max\{N_1, N_2\}$, we have

$$|f_n^2(x) - f^2(x)| = |f_n(x) - f(x)||f_n(x) + f(x)| < \frac{\epsilon}{|3f(x)+1)}|2f(x) + 1 + f(x)| = \epsilon.$$

□

Solution: Consider the set $B_n = \{f \in L^1[0,1] : \|f\|_2 \leq n\}$. Then it is clear that $L^2[0,1] \subseteq \bigcup_1^\infty B_n$. Let $f \in B_n$ and $h \in L^1[0,1] \backslash L^2[0,1]$. Then the sequence $(f + \frac{1}{k}h)_1^\infty$ converges to f in L^1 as $k \to \infty$. Indeed,

$$\|f - f - \tfrac{1}{k}h\|_1 = \|\tfrac{1}{k}h\|_1 = \frac{1}{k}\|h\|_1 \to 0.$$

However, we observe that $f + \frac{1}{k}h \notin L^2[0,1]$ since $h \notin L^2[0,1]$. Thus, $h \notin B_n$ for any $n \in \mathbb{N}$, so B_n has empty interior.

Now suppose that g is a limit point of B_n, so there exists a sequence $(g_k)_1^\infty \subset B_n$ such that $g_k \to g$ in L^1 as $k \to \infty$. We know $g \in L^1[0,1]$ since $L^1[0,1]$ is a Banach space. Moreover, since $g_k \to g$ in L^1 then there exists a subsequence $(g_{k_j})_{j=1}^\infty$ such that $g_{k_j} \to g$ pointwise a.e., hence $g_{k_j}^2 \to g^2$ pointwise, so by Fatou's lemma,

$$\begin{aligned}
\|g\|_2 &= \left(\int_0^1 |g|^2\right)^{1/2} \\
&= \left(\int_0^1 \liminf_{j\to\infty} |g_{k_j}|^2\right)^{1/2} \\
&\leq \liminf_{j\to\infty} \left(\int_0^1 |g_{k_j}|^2\right)^{1/2} \\
&= \liminf \|g_{k_j}\|_2 \\
&\leq n.
\end{aligned}$$

Therefore, B_n is closed for all $n \in \mathbb{N}$, so $L^2[0,1]$ is a meager subset of $L^1[0,1]$.

4.11 Holder's Inequality

Prove Holder's inequality using Young's inequality.

Tools:

- (**Young's Inequality**) If a, b are nonnegative real number s and if p, q are conjugate exponents, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where equality holds iff $a^p = b^q$.

- **Thm 6.2** (**Holder's Inequality**) Suppose p, q are conjugate exponents. If f, g are measurable functions on X, then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds above iff $\alpha|f|^p = \beta|g|^q$ a.e. for some α, β not both zero.

Solution: We first observe that if either f or g is equal to 0 a.e., then Holder's inequality holds. Similarly, it clearly holds if $\|f\|_p = \infty$ or $\|g\|_q = \infty$.

For any $x \in X$, we know by Young's inequality that

$$\frac{|f(x)|}{\|f\|_p} \frac{|g(x)|}{\|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}$$

Thus, we have that

$$\frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq \frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q^q}$$

Integrating over X, we have

$$\frac{\|fg\|_1}{\|f\|_p \|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

Hence,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

4.12 Minkowski's Inequality

Prove Minkowski's inequality using Holder's inequality.

Tools:

- **Thm 6.5** (**Minkowski's Inequality**) If $1 \leq p < \infty$ and $f, g \in L^p$, then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Solution: We first note that if $p = 1$ then

$$\int |f+g| \leq \int |f| + |g| = \int |f| + \int |g| = \|f\|_1 + \|g\|_1$$

Also, if $f + g = 0$ a.e., then the result holds. Thus, we observe that

$$\|f+g\|_p^p = \int |f+g|^p = \int |f+g| \cdot |f+g|^{p-1} \leq \int |f(f+g)^{p-1}| + \int |g(f+g)^{p-1}|$$

Applying Holder's inequality with $q = p/(p-1)$, we have

$$\begin{aligned}
\int |f(f+g)^{p-1}| + \int |g(f+g)^{p-1}| &= \|f(f+g)^{p-1}\|_1 + \|g(f+g)^{p-1}\|_1 \\
&\leq \|f\|_p \|(f+g)^{p-1}\|_q + \|g\|_p \|(f+g)^{p-1}\|_q \\
&= (\|f\|_p + \|g\|_p) \left(\int \left(|f+g|^{p-1}\right)^q \right)^{1/q} \\
&= (\|f\|_p + \|g\|_p) \|f+g\|_p^{p-1}
\end{aligned}$$

Thus, we have

$$\|f+g\|_p = \|f+g\|_p^p \|f+g\|_p^{1-p} \leq (\|f\|_p + \|g\|_p) \|f+g\|_p^{p-1} \|f+g\|_p^{1-p} = \|f\|_p + \|g\|_p$$

5 Part C Exercises

5.1 UCR RA Qual 2019

Show that the dual space of $L^\infty[0,1]$ strictly contains $L^1[0,1]$

Tools:

- **Def:** (dual space). Let X be a normed vector space and $K = \mathbb{R}$ or $\mathbb{C}$ (generally assume $\mathbb{C}$). Then the space $L(X, K)$ of bounded linear functionals from $X \to K$ is called the dual space of X and is also denoted X^*.
- **Def:** (operator norm). It is easily shown that $L(X,Y)$ is a vector space when X, Y are normed vector spaces. Then the operator norm is a norm on $L(X,Y)$ where for $T \in L(X,Y)$

$$\|T\| = \sup_{\substack{x \in X \\ \|x\|_X = 1}} \|T(x)\|_Y$$

- $L^\infty(X)$ is the set of *essentially bounded* measurable functions.

$$L^\infty(X) = \{f : X \to \mathbb{C} : f \text{ measurable}, \|f\|_\infty < \infty \ \mu\text{-a.e.}\}$$

  where $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

- $L^1(X)$ is the set of *integrable* measurable functions.

$$L^1(X) = \{f : X \to \mathbb{C} : f \text{ measurable}, \|f\|_1 < \infty\}$$

where $\|f\|_1 = \int_X |f|$.

- **Def:** (isometry). For $T \in L(X, Y)$, T is called an isometry if for all $x \in X$, $\|T(x)\|_Y = \|x\|_X$. Note that an isometry is an embedding into Y and an isomorphism onto $T(X)$, the range.

- Theorem 6.8a (Folland). If f, g are measurable functions on X, then $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$. This is directly derived from Hölder's inequality.

- Theorem 5.7 (Folland). (The Complex Hahn-Banach theorem). Let

  (i) X be a complex vector space

  (ii) ρ a seminorm on X

  (iii) $\mathcal{M}$ a subspace of X

  (iv) f a complex linear functional on $\mathcal{M}$

  such that $|f(x)| \leq \rho(x)$ for $x \in \mathcal{M}$. Then there exists a complex linear functional F on X such that $|F(x)| \leq \rho(x)$ for all $x \in X$ and $F|_{\mathcal{M}} = f$.

Solution: Since the dual space of $L^\infty[0,1]$ cannot literally contain $L^1[0,1]$, we shall show that there exists an isometry, $\Phi : L^1[0,1] \to (L^\infty[0,1])^*$, hence $\Phi(L^1[0,1]) \subseteq (L^\infty[0,1])^*$, and then we shall show that there exists $\mathcal{F} \in (L^\infty[0,1])^*$ such that $\mathcal{F} \neq \Phi(g)$ for any $g \in L^1[0,1]$.

Fix $f \in L^1[0,1]$ and define $\Phi : L^1[0,1] \to (L^\infty[0,1])^*$ by $\Phi(f) = \Phi_f$ where $\Phi_f : L^\infty[0,1] \to \mathbb{C}$ and

$$\Phi_f(g) = \int_0^1 fg$$

We'll first show that Φ_f is indeed a bounded linear functional on $L^\infty[0,1]$. It is clear that Φ_f is linear since the integral is a complex linear functional on the space of complex valued integrable functions. Next, by theorem 6.8a, observe that

$$\begin{aligned}
\|\Phi_f\| &= \sup_{\substack{g \in L^\infty[0,1] \\ \|g\|_\infty = 1}} |\Phi_f(g)| \\
&= \sup_{\substack{g \in L^\infty[0,1] \\ \|g\|_\infty = 1}} \left| \int_0^1 fg \right| \\
&\leq \sup_{\substack{g \in L^\infty[0,1] \\ \|g\|_\infty = 1}} \|fg\|_1 \\
&\leq \sup_{\substack{g \in L^\infty[0,1] \\ \|g\|_\infty = 1}} \|f\|_1\|g\|_\infty \\
&= \|f\|_1 \\
&< \infty
\end{aligned}$$

Thus, $\Phi_f \in (L^\infty[0,1])^*$.

Next, in order to prove that Φ is an isometry, we'll show that $\|\Phi(f)\| = \|f\|_1$. By the calculation above, it suffices to show that $\|\Phi(f)\| \geq \|f\|_1$. To do this, let us consider the function $\overline{\text{sgn}(f)}$ where sgn is the complex sign function $\text{sgn}(z) = z/|z|$ for $z \neq 0$ and $\text{sgn}(z) = 0$ otherwise. Then, it is clear that

$$\left\|\overline{\text{sgn}(f)}\right\|_\infty = \left\|\frac{\overline{f}}{|f|}\right\|_\infty = \sup_{x\in(0,1]} \left|\frac{\overline{f(x)}}{|f(x)|}\right| = \sup_{x\in(0,1]} \frac{\left|\overline{f(x)}\right|}{|f(x)|} = 1$$

Hence, $\overline{\text{sgn}(f)} \in L^\infty[0,1]$. Now we see that

$$\begin{aligned} \|\Phi_f\| &= \sup_{\substack{g\in L^\infty[0,1] \\ \|g\|_\infty = 1}} |\Phi_f(g)| \\ &= \sup_{\substack{g\in L^\infty[0,1] \\ \|g\|_\infty = 1}} \left|\int_0^1 fg\right| \\ &\geq \left|\int_0^1 f\cdot\overline{\text{sgn}(f)}\right| \\ &= \left|\int_0^1 \frac{f\cdot\overline{f}}{|f|}\right| \\ &= \int_0^1 |f| \\ &= \|f\|_1. \end{aligned}$$

Thus, Φ is an isometry from $L^1[0,1]$ to $(L^\infty[0,1])^*$.

To show that $(L^\infty[0,1])^* \not\subseteq \Phi(L^1[0,1])$, first consider $f_n : [0,1] \to \mathbb{C}$ by

$$f_n(x) = \max\{1 - nx, 0\}$$

Then $f_n \in L^\infty[0,1]$ for all $n \in \mathbb{N}$. For every $g \in L^1[0,1]$ we have that $\lim_{n\to\infty}(gf_n) = 0$ a.e. and $|gf_n| \leq |g| \in L^1[0,1]$, so by the Dominated Convergence theorem,

$$\lim_{n\to\infty} \Phi_g(f_n) = \lim_{n\to\infty} \int_0^1 gf_n = \int_0^1 0 = 0$$

Now define $\mathcal{F} : C[0,1] \to \mathbb{C}$ by $\mathcal{F}(f) = f(0)$, where $C[0,1]$ is the space of continuous complex-valued functions on $[0,1]$. $\mathcal{F}$ is well defined since if $f, g \in C[0,1]$ with $f = g$ a.e., then $f = g$ for all $x \in [0,1]$ since if there exists an $x_0 \in [0,1]$ such that $f(x_0) \neq g(x_0)$ then since f, g are continuous, there must exist some neighborhood about x_0 such that $f \neq g$ differ completely, hence contradicting that $f = g$ a.e. Thus, $\mathcal{F}(f) = \mathcal{F}(g)$. Moreover, it is clear that $C[0,1]$ is a complex subspace of $L^\infty[0,1]$, $\mathcal{F}$ is linear on $C[0,1]$ and $|\mathcal{F}(f)| \leq \|f\|_\infty$ for all $f \in C[0,1]$. Then by the Hahn-Banach theorem, there exists $\tilde{\mathcal{F}} \in (L^\infty[0,1])^*$ such that $|\tilde{\mathcal{F}}(f)| \leq \|f\|_\infty$ for all $f \in L^\infty[0,1]$ and $\tilde{\mathcal{F}}|_{C[0,1]} = \mathcal{F}$.

Thus, we see that

$$\lim_{n\to\infty} \tilde{\mathcal{F}}(f_n) = \lim_{n\to\infty} \mathcal{F}(\max\{1-nx,0\}) = \lim_{n\to\infty} \max\{1-nx,0\}|_{x=0} = 1.$$

Hence,

$$\lim_{n\to\infty} \tilde{\mathcal{F}}(f_n) = 1 \neq 0 = \lim_{n\to\infty} \Phi_g(f_n) \qquad \text{for all } g \in L^1[0,1],$$

so $\tilde{\mathcal{F}}$ does not correspond to any $g \in L^1[0,1]$. Hence $(L^\infty[0,1])^* \not\subseteq \Phi(L^1[0,1])$. ■

5.2 UCR 209C 2021 Final

Prove theorem 1.40 (Papa Rudin): Let $(X, \mathcal{M}, \mu)$ be a measure space and μ be a finite, positive measure. Consider a measurable function $f : X \to \mathbb{C}$. Suppose for every $E \in \mathcal{M}$ with $\mu(E) > 0$ it holds that the average

$$\frac{1}{\mu(E)} \int_E f(x)dx \in B$$

where B is a closed subset of $\mathbb{C}$. Show that $f(x) \in B$ for μ-a.e. $x \in X$.

Solution: By contraposition, suppose that $f(x) \notin B$ on a nontrivial set. In other words,

$$\mu(f^{-1}(B^c)) > 0.$$

Since $B \subseteq \mathbb{C}$ is closed, then B^c is open in $\mathbb{C}$, so $B^c = \bigcup_1^\infty B_{r_i}(z_i)$ where $r_i > 0$, $z_i \in \mathbb{C}$ and since $\mu(f^{-1}(B^c)) > 0$, then there exists some $N \in \mathbb{N}$ such that $\mu(f^{-1}(B_{r_N}(z_N))) > 0$. Now let $E = f^{-1}(B_{r_N}(z_N))$. We know that $E \in \mathcal{M}$ since $B_{r_N}(z_N)$ is open so it is a Borel set in $\mathbb{C}$ and f is measurable, so $f^{-1}(E) \in \mathcal{M}$. Then in order to show that

$$\frac{1}{\mu(E)} \int_E f(x)dx \in B_{r_N}(z_N) \subseteq B^c$$

we observe that

$$\begin{aligned}
\left| \frac{1}{\mu(E)} \int_E f(x)dx - z_N \right| &= \left| \frac{1}{\mu(E)} \int_E f(x)dx - \frac{1}{\mu(E)} \int_E z_N \right| \\
&= \frac{1}{\mu(E)} \left| \int_E f(x) - z_N dx \right| \\
&\leq \frac{1}{\mu(E)} \int_E |f(x) - z_N| dx \\
&< \frac{1}{\mu(E)} \int_E r_N dx \\
&= r_N
\end{aligned}$$

as desired. ■

5.3 Folland 5.55a

Let $\mathcal{H}$ be a Hilbert space. Prove the polarization identity: for any $x, y \in \mathcal{H}$,

$$\langle x, y \rangle = \frac{1}{4} \left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \right)$$

Tools:

- **Def.** (inner product) Let $\mathcal{H}$ be a complex vector space ($\mathbb{C}$ is the underlying field). An inner product on $\mathcal{H}$ is a map $(x, y) \mapsto \langle x, y \rangle$ from $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that :

  (i) $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for all $x, y, z \in \mathcal{H}$ and $a, b \in \mathbb{C}$.

(ii) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for all $x, y \in \mathcal{H}$

(iii) $\langle x, x\rangle \in (0, \infty)$ for all nonzero $x \in \mathcal{H}$.

- $\text{Re}(-i(a+bi)) = \text{Re}(b - ai) = \text{Im}(a+bi)$

Solution: By the definition of inner products (i,ii) and the norm induced by it ($\|x\| = \sqrt{\langle x, x\rangle}$), we see that

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
-\|x-y\|^2 &= -\langle x-y, x-y\rangle = -\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle - \langle y, y\rangle \\
i\|x+iy\|^2 &= i\langle x+iy, x+iy\rangle = i\langle x, x\rangle + i\langle x, iy\rangle + i\langle iy, x\rangle - i\langle y, y\rangle \\
-i\|x-iy\|^2 &= -i\langle x-iy, x-iy\rangle = -i\langle x, x\rangle + i\langle x, iy\rangle + i\langle iy, x\rangle + i\langle y, y\rangle
\end{aligned}$$

Taking the sum of all 4 terms gives

$$\begin{aligned}
\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 &= 2(\langle x, y\rangle + \langle y, x\rangle) + 2i(\langle x, iy\rangle + \langle iy, x\rangle) \\
&= 2(2\,\text{Re}\,\langle x, y\rangle) + 2i(2\,\text{Re}\,\langle x, iy\rangle) \\
&= 4\,\text{Re}\,\langle x, y\rangle + 2i(2\,\text{Re}(-i(\langle x, y\rangle))) \\
&= 4\,\text{Re}\,\langle x, y\rangle + 4i\,\text{Im}\,\langle x, y\rangle \\
&= 4\langle x, y\rangle \quad \blacksquare
\end{aligned}$$

5.4 UCR 209C 2021 Midterm

Construct a linear map $F : L^2(\mathbb{T}, m) \to \ell^2(\mathbb{Z})$ such that F is surjective and it preserves the inner products. Show that it has each property.

Tools:

- $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1]/0 \sim 1$ which is the 1-dimensional unit torus.
- $L^2(X, \mu)$ is the set of measurable complex-valued functions with bounded L^2-norm. That is, if $f \in L^2(X, \mu)$, then
$$\|f\|_2 = \left(\int_X |f|^2 d\mu\right)^{1/2} < \infty$$
- $\ell^2(\mathbb{Z})$ is the space of complex-valued sequences over the integers with bounded ℓ^2-norm. That is, if $f \in \ell^2(\mathbb{Z})$ then
$$\|f\|_{\ell^2} = \left(\sum_{n\in\mathbb{Z}} f(n)\right)^{1/2} < \infty$$
- **Def.** (Hilbert space). A complex vector space $\mathcal{H}$ equipped with an inner product $\langle \cdot, \cdot \rangle$ which is complete with respect to the norm induced by the inner product $\|x\| = \sqrt{\langle x, x\rangle}$.
- **Def.** (orthonormal set). A set $\{u_\alpha\}_{\alpha\in A} \subset \mathcal{H}$ is called orthonormal is $\|u_\alpha\| = 1$ for all $\alpha \in A$ and if $\alpha \neq \beta$, then u_α and u_β are orthogonal, i.e., $\langle u_\alpha, u_\beta\rangle = 0$.

- Theorem 5.27 (Folland). If $\{u_\alpha\}_{\alpha\in A}$ is an orthonormal set in a Hilbert space, $\mathcal{H}$, then the following are equivalent:
  - (a) (Completeness). If $\langle x, u_\alpha\rangle = 0$ for all α, then $x = 0$.
  - (b) (Parseval's identity). $\|x\|^2 = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$ for all $x \in \mathcal{H}$.
  - (c) For each $x \in \mathcal{H}$, $x = \sum_{\alpha\in A} \langle x, u_\alpha\rangle u_\alpha$, where the sum on the right has only countably many nonzero term and converges in the norm topology no matter how these terms are ordered.
- Folland 5.55a. (Polarization identity for complex vector spaces).

Solution: Let $f \in L^2(\mathbb{T})$ and recall the Fourier transform,

$$\mathcal{F}(f) = \int_{\mathbb{T}} f(x)e^{-2\pi inx}dx = \langle f, E_n\rangle$$

where $E_t = e^{2\pi inx}$. Then define $F(f) = \mathcal{F}(f)(n) = \hat{f}(n)$.

Let us first show that $F(f) \in \ell^2(\mathbb{Z})$. We first recall that $\{E_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$. Moreover, it is clear that if $f \neq 0$ m-a.e., then $\int_{\mathbb{T}} f(x)dx \neq 0$ as well. Thus, by theorem 5.27b, we have that Parseval's identity holds, so

$$\|F(f)\|^2_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \langle f, E_n\rangle = \|f\|^2_{L^2} < \infty$$

since $f \in L^2(\mathbb{T})$. Thus, $F(f) \in \ell^2(\mathbb{Z})$.

Next, we'll show that F is linear. Let $f, g \in L^2(\mathbb{T})$ and $c \in \mathbb{C}$. Then

$$F(cf + g) = \langle cf + g, E_n\rangle = c\langle f, E_n\rangle + \langle g, E_n\rangle = cF(f) + F(g)$$

by the linearity of the inner product in the first component.

For surjectivity, let $c(n) \in \ell^2(\mathbb{Z})$ and consider $f(x) = \sum_{k\in\mathbb{Z}} c(k)e^{2\pi ikx} = \sum_{k\in\mathbb{Z}} c(k)E_k$. Since $\{E_k\}_{k\in\mathbb{Z}}$ is a basis for $L^2(\mathbb{T})$, then $f(x) \in \text{span}\{E_k : k \in \mathbb{Z}\} \subset L^2(\mathbb{T})$. Now observe that by theorem 5.27c,

$$f = \sum_{n\in\mathbb{Z}} \langle f, E_n\rangle E_n = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c(k)\langle E_k, E_n\rangle E_n = \sum_{n\in\mathbb{Z}} c(n)E_n$$

since $\langle E_k, E_n\rangle = 1$ only when $k = n$. Then,

$$\sum_{n\in\mathbb{Z}} (\langle f, E_n\rangle - c(n))E_n = 0$$

so it must be that $\langle f, E_n\rangle = F(f) = c(n)$ for each $n \in \mathbb{Z}$. Hence F is surjective.

Last, we will show that F preserves inner products. Since Parseval's identity holds, we know that $\|F(f)\|^2_{\ell^2} = \|f\|^2_{L^2}$, then by the polarization identity (Folland 5.55a),

$$\begin{aligned}\left\langle \hat{f}, \hat{g}\right\rangle &= \frac{1}{4}\left(\|\hat{f}+\hat{g}\|^2_{\ell^2} + \|\hat{f}-\hat{g}\|^2_{\ell^2} + i\|\hat{f}+i\hat{g}\|^2_{\ell^2} - i\|\hat{f}-i\hat{g}\|^2_{\ell^2}\right)\\ &= \frac{1}{4}\left(\|f+g\|^2_{L^2} + \|f-g\|^2_{L^2} + i\|f+ig\|^2_{L^2} - i\|f-ig\|^2_{L^2}\right)\\ &= \langle f, g\rangle \quad \blacksquare\end{aligned}$$

5.5 UCR RA Qual 2017

Show that $L^\infty[0,1]$ is not separable, i.e. that is does not have a countable dense subset.

Solution: Suppose by contradiction that $L^\infty[0,1]$ is separable, so there exists a countable dense set $D \subset L^\infty[0,1]$. Consider the family of functions $\{\chi_r\}_{r\in[0,1]} \subset L^\infty[0,1]$ where χ_r is the characteristic function on $\{r\}$. Then it is clear that for $r, r' \in \mathbb{R}$ where $r \neq r'$ we have that $\|\chi_r - \chi_{r'}\|_\infty = 1$. Since D is dense, then there exists $f \in D$ such that $\|f - \chi_r\|_\infty < \frac{1}{2}$. Now notice,

$$1 = \|\chi_r - \chi_{r'}\|_\infty \leq \|\chi_r - f\|_\infty + \|f - \chi_{r'}\|_\infty < \frac{1}{2} + \|f - \chi_{r'}\|_\infty$$

Thus, $\|f - \chi_{r'}\|_\infty > \frac{1}{2}$, so χ_r is the only function in $\{\chi_r\}_{r\in[0,1]}$ such that f is within $\frac{1}{2}$ distance to it in the L^∞-metric. Hence, we may define a mapping $r \mapsto f$ which is injective by above. This contradicts that $[0,1]$ is uncountable since D is dense. ■

5.6 UCR 209C 2021 HW

Show that the metric of $L^\infty(X, d\mu)$ induced by the norm of L^∞ is complete.

Solution: Let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $L^\infty(X, d\mu)$ and let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$

$$\|f_n - f_m\|_\infty < \frac{\epsilon}{2}$$

Next, there exists a sequence of numbers $(n_k)_{k\geq 1}$ such that

$$|(f_{n_k+1} - f_{n_k})(x)| \leq \|f_{n_k+1} - f_{n_k}\|_\infty < 2^{-k}$$
$$\sum_{k=1}^{\infty} |(f_{n_k+1} - f_{n_k})(x)| \leq \sum_{k=1}^{\infty} \|f_{n_k+1} - f_{n_k}\|_\infty < 1$$

Thus, each series above converges absolutely, so we know

$$f(x) := f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_k+1} - f_{n_k})(x) = \lim_{k\to\infty} f_{n_k}$$

exists. Then,

$$\left|(\lim_{k\to\infty} f_{n_k} - f_n)(x)\right| = \lim_{k\to\infty} |(f_{n_k} - f_n)(x)| \leq \lim_{k\to\infty} \|f_{n_k} - f_n\|_\infty < \lim_{k\to\infty} \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

for all $n_k, n \geq N$. Thus, $|(f - f_n)(x)| < \frac{\epsilon}{2}$ for $n \geq N$, so $\|f - f_n\|_\infty \leq \frac{\epsilon}{2} < \epsilon$. Therefore, $\|\cdot\|_\infty$ is complete.

5.7 UCR 209C 2021 HW

Let $\phi(x) = \frac{1}{2}e^{-|x|}$ on $\mathbb{R}$. Compute $\mathcal{F}(\phi)$. Then use the Fourier transformation to show that $u = f * \phi$ solves the ODE

$$u - u'' = f$$

Tools:

- **Thm 8.26** (**The Fourier Inversion Theorem**) If $f \in L^1$, we define

$$\mathcal{F}^{-1}(f)(x) = \hat{f}(-x) = \int f(\xi)e^{2\pi i\xi x}d\xi.$$

  if $\hat{f} \in L^1$ as well, then f agrees almost everywhere with a continuous function f_0, and $\mathcal{F}^{-1}(\hat{f}) = \hat{(\mathcal{F}^{-1}(f))} = f_0$.

- **Thm 2.37** (**The Fubini-Tonelli Theorem**) Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces.

  (a) (Tonelli) If $f \in L^+(X \times Y)$, then the functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^+(X)$ and $L^+(Y)$, respectively, and

$$\begin{aligned}\int f d(\mu \times \nu) &= \int \left[\int f(x,y)d\nu(y)\right] d\mu(x)\\ &= \int \left[\int f(x,y)d\mu(x)\right] d\nu(y)\end{aligned}$$

  (b) (Fubini) If $f \in L^1(\mu \times \nu)$, then $f_x \in L^1(\nu)$ for a.e. $x \in X$, $f^y \in L^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x d\nu$ and $h(y) = \int f^y d\mu$ are in $L^1(\mu)$ and $L^1(\nu)$ respectively and the integral equality of Tonelli's holds as well.

Solution: Calculating the Fourier transformation of ϕ we have,

$$\begin{aligned}\mathcal{F}(\phi)(t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi(x)e^{-itx}dx\\ &= \frac{1}{2\sqrt{2\pi}}\int_{\mathbb{R}} e^{-|x|}e^{-itx}dx\\ &= \frac{1}{2\sqrt{2\pi}}\left(\int_{-\infty}^{0} e^{(1-it)x}dx + \int_{0}^{\infty} e^{-(1+it)x}dx\right)\\ &= \frac{1}{2\sqrt{2\pi}}\left(\frac{1}{1-it}e^{(1-itx)}\Big|_{-\infty}^{0} - \frac{1}{1+it}\Big|_{0}^{\infty}\right)\end{aligned}$$

Since

$$|e^x e^{-itx}| = |e^x||\cos(tx) - i\sin(tx)| = |e^x|$$

then we may evaluate the limits above to get

$$\mathcal{F}(\phi)(t) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1-it} + \frac{1}{1+it}\right) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+t^2}$$

First recall that

$$\mathcal{F}(u') = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} u'(x)e^{-itx}dx$$

For $u = e^{-itx}$ and $dv = u'(x)$, we have by integration by parts

$$\begin{aligned}\mathcal{F}(u')(t) &= \frac{1}{\sqrt{2\pi}}\left(u(x)e^{-itx}\Big|_{-\infty}^{\infty} + (it)\int_{\mathbb{R}} u(x)e^{-itx}dx\right) \qquad (*)\\ &= (it)\mathcal{F}(u)(t)\end{aligned}$$

Next, recall that

$$\begin{aligned}\mathcal{F}(f*\phi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(f*\phi)(x)e^{-itx}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x-y)\phi(y)dy\right)e^{-itx}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x-y)\phi(y)dy\right)e^{-itx}e^{-ity}e^{ity}dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x-y)\phi(y)dy\right)e^{-it(x-y)}e^{-ity}dx \qquad (**)\\ &= \sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x-y)e^{-it(x-y)}dx\right)\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\phi(y)e^{-ity}dy\right)\\ &= \sqrt{2\pi}\mathcal{F}(f)(t)\mathcal{F}(\phi)(t)\end{aligned}$$

Now in order to show that $u = f*\phi$ solves the ODE $u - u'' = f$ above, we take Fourier tranforms of both sides.

$$\begin{aligned}\mathcal{F}(f) &= \mathcal{F}(u-u'')\\ &= \mathcal{F}(u) - \mathcal{F}(u'')\\ &= \mathcal{F}(f*\phi) - (it)^2\mathcal{F}(f*\phi)\\ &= \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(\phi) + t^2\sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(\phi)\\ &= \sqrt{2\pi}(1+t^2)\mathcal{F}(f)\mathcal{F}(\phi)\\ &= \frac{1}{\mathcal{F}(\phi)}\mathcal{F}(f)\mathcal{F}(\phi)\\ &= \mathcal{F}(f).\end{aligned}$$

Hence $u = f*\phi$ does satisfy $u - u'' = f$.

For the assumptions on f, we first note that in order to fully solve the ODE, we need to use the inverse Fourier transform, so by the Fourier inversion theorem, we require that $f, \mathcal{F}(f) \in L^1(\mathbb{R})$ and so that $\mathcal{F}^{-1}$ may be taken. Next, in $(*)$, we made use of the fact that the boundary term from integration by parts disappeared. In order to make such an assumption, we require that $f \in C^2(\mathbb{R})$ so that we may take the second derivative, $f^{(k)} \in L^1(\mathbb{R})$ for $k = 0, 1, 2$ so that we may integrate and $f^{(k)} \in C_0$ for $k = 0, 1$ so that the boundary term (after differentiating) disappears. Last, we used Fubini's theorem in $(**)$, which requires that $f \in L^1(\mathbb{R}\times\mathbb{R})$ when written in terms of x and y but this follows from $f \in L^1(\mathbb{R})$.

5.8 UCR RA Qual 2016

Let $f(x) = \frac{1}{2} - x$ on the interval $[0, 1)$. Extend f to be periodic on $\mathbb{R}$ and use the Fourier series of f to show that

$$\sum_{k \geq 1} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Tools:

- The fourier series of $f : \mathbb{T} \to \mathbb{C}$ is

$$\sum_{k \in \mathbb{Z}} \hat{f}(k) E_k = \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{T}} f(x) e^{-2\pi i k x} dx \right) e^{-2\pi i k x}$$

Solution: First, we see that the Fourier transform of f when $k \neq 0$ is

$$\begin{aligned}
\hat{f}(k) &= \int_{\mathbb{T}} (\tfrac{1}{2} - x) e^{-2\pi i k x} dx \\
&= \frac{1}{2} \int_{\mathbb{T}} e^{-2\pi i k x} dx - \int_{\mathbb{T}} x e^{-2\pi i k x} dx \\
&= \frac{1}{2} \int_{\mathbb{T}} e^{-2\pi i k x} dx + \frac{x}{2\pi i k} e^{-2\pi i k x} \Big|_0^1 + \int_{\mathbb{T}} e^{-2\pi i k x} dx \\
&= \frac{3}{2} \left(\frac{-1}{2\pi i k} \right) e^{-2\pi i k x} \Big|_0^1 + \frac{1}{2\pi i k} \\
&= \frac{1}{2\pi i k}
\end{aligned}$$

For $k = 0$, we know that $\hat{f}(0) = 0$. Thus,

$$\begin{aligned}
\sum_{k=-\infty, k \neq 0}^{\infty} \hat{f}(k) E_k = \sum_{k=-\infty, k \neq 0}^{\infty} \left(\frac{1}{2\pi i k} \right) e^{-2\pi i k x} &= \sum_{k=-\infty, k \neq 0}^{\infty} \left(\frac{1}{2\pi i k} \right) [\cos(2\pi k x) - i \sin(2\pi k x)] \\
&= \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{\pi i k}
\end{aligned}$$

5.9 UCR RA Qual 2016

Prove that if $\mathcal{H}$ is a Hilbert space, $M \subseteq \mathcal{H}$ is a closed linear subspace, and $v \in \mathcal{H}$, then there exists a point $x \in M$ achieving the minimum distance to v. In other words, if $y \in H$, then $\|y - v\| \geq \|x - v\|$.

Tools:

- **Thm 5.22** (The Parallelogram Law) For all $x, y \in \mathcal{H}$, $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Solution: Let $v \in \mathcal{H}$, $\delta = \inf\{\|x - v\| : x \in M\}$, and $\epsilon > 0$. Then there exists a sequence $(x_n)_1^\infty \subset M$ such that $\|x_n - v\| \to \delta$ as $n \to \infty$. Let $\epsilon > 0$. We know by the parallelogram law that

$$2\|x_n - v\|^2 + 2\|x_m - v\|^2 = \|x_n - x_m\|^2 + \|x_n + x_m - 2v\|^2$$

so

$$\begin{aligned}\|x_n - x_m\|^2 &= 2\|x_n - v\|^2 + 2\|x_m - v\|^2 - \|x_n + x_m - 2v\|^2 \\ &= 2\|x_n - v\|^2 + 2\|x_m - v\|^2 - 4\|\tfrac{1}{2}(x_n + x_m) - v\|^2\end{aligned}$$

Since M is a linear space, then we know $\frac{1}{2}(x_n + x_m) \in M$, so $\|\frac{1}{2}(x_n + x_m) - v\| \geq \delta$. Moreover, since $\|x_n - v\| \to \delta$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|x_n - v\| < \delta + \epsilon$. Thus, for such n, we see

$$\|x_n - x_m\|^2 \leq 4(\delta + \epsilon)^2 - 4\delta^2 = 8\delta\epsilon + 4\epsilon^2$$

Thus, $(x_n)_1^\infty$ is Cauchy in $\mathcal{H}$ so it converges to some $x \in \mathcal{H}$, but since M is closed, then $x \in M$ and x is the element of least distance to v.

5.10 UCR Math209C 2021 HW

Prove the following version of the Riemann-Lebesgue lemma. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Show that the Fourier coefficients $\hat{f} : \mathbb{Z} \to \mathbb{C}$ of $f \in L^1(\mathbb{T}, m)$, $\hat{f}(n) = \int_{\mathbb{T}} f(x)e^{-2\pi inx}dx$ satisfy

$$\lim_{|n| \to \infty} |\hat{f}(n)| = 0.$$

You may use the fact that trigonometric polynomials are dense in $(L^1(\mathbb{T}, m), \|\cdot\|_1)$

Tools:

- If $f(x) : \mathbb{T} \to \mathbb{C}$ is a trigonometric polynomial, that is $f(x) = \sum_{k=-M}^{M} a_k e^{2\pi ikx}$, then $\|f\|_1 = 0$ when $k \neq 0$.

  Proof. It suffices to show this for a single term

$$\begin{aligned}\|E_k\|_1 &= \int_{\mathbb{T}} a_k e^{2\pi ikx} dx \\ &= a_k \frac{1}{2\pi ik}[\cos(2\pi kx) + i\sin(2\pi kx)]_0^1 \\ &= a_k \frac{1}{2\pi ik}(1 - 1) \\ &= 0\end{aligned}$$

□

Solution: Since trigonometric polynomials are dense in $L^1(\mathbb{T})$ then there exists $h = \sum_{k=-M}^{M} a_k e^{2\pi ikx}$, $a_k \in \mathbb{C}$ such that $\|h - f\|_1 < \epsilon$

$$\begin{aligned}
|\hat{f}(n)| &= \left| \int_{\mathbb{T}} f(x) e^{-2\pi inx} dx \right| \\
&\leq \int_{\mathbb{T}} (|f(x) - h(x)| + |h(x)|)|e^{-2\pi inx}| dx \\
&= \|f - h\|_1 + \|hE_n\|_1 \\
&< \epsilon + \left\| \sum_{k=-M}^{M} E_{n-k} \right\|
\end{aligned}$$

And since $\|E_{n-k}\| = 0$ when $n \neq k$ then we may choose $|n| > M$ so that,

$$|\hat{f}(n)| < \epsilon$$

Thus, $\lim_{|n|\to\infty} \hat{f}(n) = 0$.